Assessment of Control Loop Performance

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For processes described by linear transfer functions with additive disturbances, the best possible control in the mean square sense is realized when a minimum variance controller is implemented. It is shown that an estimate of the best possible control can be obtained by fitting a univariate time series to process data collected under routine control. No ‘identifiability’ constraints need be imposed. The use of this technique is demonstrated with pilot plant and production data.

Pour les procédés décrits par des fonctions de transfert linéaires avec des perturbations additives, le meilleur contrôle possible se traduit par un régulateur à variance minimum. On montre qu’on peut obtenir une estimation du meilleur contrôle possible en adaptant une série chronologique univariée pour traiter des données recueillies lors d’un rétrocontrôle de routine. Il n’est pas nécessaire d’imposer des contraintes d’identifiabilité. L’utilisation de cette technique est décrite à partir de données d’une usine pilote et des données de production.

Keywords: minimum variance control, autocorrelation, time series analysis, control loop performance.

There are many techniques available for the design of feedback control strategies. These methods may be very simple, such as those used in statistical process control, or they may be more elaborate strategies, such as pole placement designs and linear quadratic controllers. Regardless of the control strategy, it is important to have some benchmark against which its performance can be evaluated. The theoretical ‘best achievable’ control, as measured by the mean square error, is such a benchmark. If the theoretical best achievable control represents a significant improvement over the current performance, alternate controller tuning or feedback control strategies can be considered if this improved performance is warranted. However, the best achievable performance itself may not be adequate. In these cases alternative approaches, such as feedforward control, reduction of deadtime and different loop pairings must be used to achieve a reduction in variability.

The purpose of this paper is to describe a very simple technique for ascertaining the best theoretically achievable feedback control performance as measured by the output mean square error. A univariate time series model is fit to process data. From this time series model, the best control performance can be estimated if the number of whole periods of delay is known. An important feature of this method is that it is not necessary to ‘perturb’ the process with extraneous test signals. Data may be used from any linear process which is operating under normal closed loop control with a linear, time-invariant feedback control strategy.

The outline of this paper is as follows. In the first section the process description is given. This is followed by a brief derivation of a minimum variance controller, and of the resulting properties of a process operating under minimum variance control. It is then shown that the theoretical best achievable performance in the mean square sense can be estimated from process data collected under ‘normal’ closed loop conditions. A method for estimating this mean square performance is discussed. This is followed by a Monte-Carlo simulation to illustrate the statistical properties of the estimator. Finally, this technique is applied to pilot plant and production data.

Minimum Variance Control

The process is described by the superposition of a discrete dynamic model relating the controlled variable \( Y_t \), to the manipulated variable \( U_t \), and a disturbance \( D_t \).

\[
Y_t = w(z^{-1})/\delta(z^{-1})U_{t-b} + D_t \quad \ldots (1)
\]

\( w(z^{-1}) \) and \( \delta(z^{-1}) \) are polynomials of order \( (r,s) \) respectively in the backward shift operator \( z^{-1} \), which is defined such that \( z^{-1}Y_t = Y_{t-1} \), \( b \) is the number of whole periods of delay in the process and is calculated as

\[
b = 1 + f + \text{integer } (\tau_d/T) \quad \ldots (2)
\]

\( T \) is the control interval and \( \tau_d \) is the process delay arising from true process deadtime or analysis delay. \( f \) is the number of integer periods of delay. Only processes which are open loop stable are considered.

The process disturbance \( D_t \), is represented by an Autoregressive-Integrated-Moving-Average (ARIMA) time series model of order \( (p,d,q) \), i.e., a linear filter driven by white noise:

\[
D_t = \theta(z^{-1})u_t/[\phi(z^{-1}) \circ v^d] \quad \ldots (3)
\]

\( \{u_t\} \) is a sequence of independently and identically distributed normal random variables with mean zero and constant variance. \( v \) is an abbreviation for \( 1 - z^{-1} \). The monic polynomials \( \phi(z^{-1}) \) and \( \theta(z^{-1}) \) are of order \( p \) and \( q \) respectively, and are stable. The presence of \( d > 0 \) roots on the unit circle allows for nonstationary behavior in the mean value of the disturbance. The observations are collected at equispaced control intervals. \( \{Y_t, U_t\} \) represent deviations of the process and manipulated variables from some reference values. The reference value for \( \{Y_t\} \) is taken as the target value or setpoint for closed loop identification, and is taken as the mean value in open loop situations.

Time series models provide a very flexible means of modeling process dynamics and disturbances. These disturbances may be of a meandering, drifting type nature, or they may be more deterministic in nature, such as randomly occurring steps, ramps, or exponential rises to a new level (MacGregor et al., 1984; Astrom and Wittenmark, 1984).

When the model Equations (1) and (2) are known, a controller can be designed to minimize the variance of \( Y_t \). (Recall that \( Y_t \) is the deviation of the process variable from its setpoint). The resulting linear controller is (Astrom, 1970; Box and Jenkins, 1970):

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The relationship between minimum variance controllers and Proportional-Integral-Derivative (PID) controllers and other strategies such as Dahlin's algorithm are discussed by Palmor and Shinnar (1979) and by Harris et al. (1981). The relationship between minimum variance controllers and methods used in statistical process control is discussed by Box et al. (1974). In these papers, it is shown that many feedforward control algorithms have the same structure as minimum variance controllers. Although a particular controller, such as a PID algorithm, may not have been designed to explicitly minimize the variance of the output, it may still give performance which is very close to that of the minimum variance controller. This is readily verified by examining the autocorrelation function of the process variable. The autocorrelation function of the data, commonly known as the correlogram, and the Fourier transform of the autocorrelation function, which is the spectrum of the data, also provide valuable information for diagnosing the sources of process disturbances and diagnosing control loop performance (Pryor, 1982; Ohtsu and Kitagawa, 1984; DeVries and Wu, 1978).

It may not be possible to implement a minimum variance controller due to the fact that (i) excessive control action is required or (ii) the numerator of the process transfer function \( w(z^{-1}) \) has a zero, whose magnitude, in the \( z \)-plane exceeds one. The former typically arises when: i) the sampling interval is short relative to the dominant process time constants or ii) the fractional period of delay is significant. In such cases, constraining the control effort is required, which in turn inflates the output variance. However, the variance estimate given by Equation (10) provides a theoretical lower bound on the output variance without linear feedback control.

**Estimation of Minimum Variance Performance**

When it is concluded that the existing control strategy is not giving minimum variance performance, it is of interest then to have a technique for estimating the best possible performance. Ideally, this technique should be simple. In addition, it is desirable to use data collected under conditions which represent normal or routine process operation, i.e., where there are no external inputs used to perturb the process. The following analysis shows when it is possible to do this.

Let the process be controlled with a linear, time-invariant controller, having the transfer function \( G_c(z^{-1}) \), i.e.,

\[
U_t = -G_c(z^{-1})Y_t
\]

Substituting this equation into the process description, Equation (1), and using the identity (6) gives

\[
Y_t = \psi(z^{-1})a_t + \frac{C(z^{-1})}{D(z^{-1})} Y_{t-f-1}
\]

where

\[
C(z^{-1}) = T(z^{-1})\delta(z^{-1}) - w(z^{-1}) \psi(z^{-1}) \upsilon^d
\]

\[
\phi(z^{-1}) = \theta(z^{-1}) \upsilon^d
\]

and

\[
D(z^{-1}) \delta(z^{-1})
\]

Alternatively, the transfer function between \( Y_t \) and \( a_t \) can be expressed as:

\[
U_t = -G_c(z^{-1})Y_t
\]
The polynomials \( F(z^{-1}) \) and \( G(z^{-1}) \) follow from the Equations (13)–(15).

The first term on the right hand side of Equation (12) is independent of the second term in this expression, since the latter only involves values of \( Y_{r-j} \) or equivalently \( a_{r-j} \), for \( j > f \). The key feature to note from Equation (12) is that the first \( f \) moving average terms of the closed loop transfer function are not affected by any form of feedback. The invariance of these parameters to feedback is simply a recognition that a feedback control strategy, linear or nonlinear, cannot return the process output to its target value until the process deadtime or time delay has elapsed. The significance of this result is that it is not necessary to jointly estimate the parameters of the process dynamics and disturbances. It is only necessary to determine the transfer function of the closed loop system, i.e., Equation (15). The order and parameters of this transfer function can be uniquely determined from a time series fit only to the \( Y \)'s. Consequently, i) it is not necessary to perturb the manipulated variable to obtain information about the process dynamics and ii) it is not necessary to insist on any ‘identifiability’ restrictions (Box and MacGregor, 1976), as is usually the case for closed loop identification. If a nonlinear controller is used, as is the case in some statistical control methods, then it is necessary to simultaneously estimate the parameters of the process dynamics and disturbances.

To estimate the theoretically achievable minimum variance performance, it is necessary to estimate the \( f+1 \) step ahead prediction error variance of the process. This requires determining the order and estimating the parameters of the polynomials \( F(z^{-1}) \) and \( G(z^{-1}) \) in Equation (15). \( f \) must also be known. Methods for accomplishing this are well known (Box and Jenkins, 1970; Newton, 1988). Typically, the parameters of the polynomials are estimated by maximizing the conditional likelihood function of the observations. The prediction error variance is estimated by

\[
\hat{\sigma}_e^2 = \hat{\sigma}_e^2 (1 + \hat{\psi}_1^2 + \ldots + \hat{\psi}_f^2) \tag{16}
\]

In Equation (16) \( \hat{\sigma}_e^2 \) is the maximum likelihood estimate of the one step ahead prediction error variance and the \( \hat{\psi} \)'s are the estimates of the moving average coefficients obtained by dividing the denominator of the identified transfer function into the numerator. This is equivalent to solving the following Diophantine equation for \( \hat{\psi}(z^{-1}) \) and \( \hat{T}(z^{-1}) \):

\[
\hat{F}(z^{-1}) = \hat{\psi}(z^{-1})\hat{G}(z^{-1}) + z^{-(f+1)} \hat{T}(z^{-1}) \tag{17}
\]

The circumslices in Equation (17) emphasize that the quantities are estimated. The estimate of the prediction error variance given by Equation (16) does not take into account the uncertainty associated with the fact that the parameters are not exact, but rather are estimates of the parameters. Properties of predictors for time series which employ estimated parameters have been investigated by Box and Jenkins (1970), Baillie (1979) and Fuller and Hasza (1981). Their results indicate that the prediction error variance is inflated by the use of estimated parameters. However, this variance inflation factor is inversely proportional to the number of observations used to fit the time series.

DeVries and Wu (1978) proposed that a time series be fit to the \( Y \)'s and \( \hat{\sigma}_a \) be used as a performance measure. \( \hat{\sigma}_a \) is an estimate of the best achievable control only when \( f = 0 \). \( \hat{\sigma}_a \) does however provide a lower bound on the achievable performance of the control strategy if it were possible to eliminate the process time delays.

**Applications**

The sampling properties of the \( f+1 \) step ahead estimate for the prediction error are first investigated via simulation. This is followed by applications of this method to pilot plant and production data.

**Sampling Properties of the Estimators**

In this section, the sampling properties of the \( f+1 \) step ahead prediction error variance, given by Equation (16), are investigated via simulation. The process description is:

\[
Y_t = \frac{14.647}{(1 - 0.35z^{-1})} U_{t-4} + \frac{1}{(1 - 1.35z^{-1} + 0.35z^{-2})} a_t \tag{18}
\]

\( Y_t \) represents the deviation from target value of the dry basis weight on a paper machine. \( U_i \) is the deviation thick stock flow rate. Model identification, development of the minimum variance controller and industrial implementation of this controller are discussed in Astrom (1970) and Priestley (1981).

The simulated process was controlled with a discrete proportional integral controller of the form:

\[
\nu U_t = -0.13Y_t + 0.011Y_{t-1} \tag{19}
\]

Four hundred observations were simulated. The \( \{a_t\}'s \) were normally distributed, with mean zero and standard deviation 0.257. The closed loop was modeled as an autoregressive process of order \( n \), i.e. \( F(z^{-1}) = 1 \). The order \( n \) was determined using the final prediction error concept of Akaike (1969). The \( f+1 \) step ahead prediction error was calculated from Equations (16) and (17). This procedure was repeated five hundred times with a different set of random numbers used in every simulation. A histogram of \( \sigma_{f+1}/\sigma_{f+1} \) is shown in Figure 1 for these simulations. From the analysis of this and other simulations, one finds that in order to obtain a good estimate of the prediction error variance, the number of observations needs to be one hundred to one hundred and fifty times \( f+1 \). Comparable results were obtained by modelling the closed loop as an ARIMA \((n, 0, n-1)\) process.

**Polymer Production Data**

Series \( L \) in Box and Jenkins (1970) is a collection of 312 hourly observations of a measure of melt viscosity of a polymer from its target value. This data is from an industrial process and was collected under closed loop conditions with no extraneous perturbations of the input.
POLYMER PRODUCTION DATA

signal. This data is plotted in Figure 2. The standard deviation of the process variable is 4.74. The autocorrelation function of the data, along with its approximate ninety-five percent confidence intervals are plotted in Figure 3. For this process, it is known that \( f = 0 \). Thus, we reject the hypothesis that the process is operating under minimum variance control.

This data was fit and adequately modeled by an ARIMA (6,0,0) and and ARIMA (2,0,1) model. Figure 4 shows the theoretical autocorrelation function for the parameters of the ARIMA (2,0,1) model used to represent this time series. This matches very closely the observed autocorrelation function, Figure 3. The estimate of \( \hat{\sigma}_1 = 4.08 \) is an estimate of the lower bound on the achievable performance if a minimum variance controller were to be implemented.

From Figure 2, we note that observations 162 and 265 might be outliers. These values were replaced by the average of their two nearest neighbours. The process standard deviation was recalculated as 4.44. The orders and parameters of the final time series were essentially unchanged. \( \hat{\sigma}_1 \) was estimated as 3.86.

PILOT PLANT REACTOR DATA

The application of self-tuning regulators for the control of a pilot plant reactor carrying out the hydrogenolysis of \( n \)-butane is discussed in Harris et al. (1980). Figure 5A shows the process variable — the hot spot temperature and the manipulated variable — the butane flowrate to the reactor. A proportional integral controller was used to regulate the hot spot temperature. The sample autocorrelation function of the hot spot temperature is shown in Figure 6A. For this process it is known that \( f = 0 \). It is clear that this process is not under minimum variance control. The hot spot temperature was adequately represented by an AR(2) model with complex roots close to the stability boundaries. Using this model \( \hat{\sigma}_1 \) was estimated as 1.95. Figure 5B shows the process variable and the manipulated variables when an adaptive controller was used to regulate the hot spot temperature. The sample autocorrelation function of the hot spot temperature,
Figure 5 — Pilot Plant Reactor Data
A: PI Controller
B: Adaptive Controller

Figure 6 — Correlogram of Pilot Plant Reactor Data
A: PI Controller
B: Adaptive Controller

Figure 6B, indicates that this is essentially minimum variance control. The standard deviation of the hot spot temperature is 1.82 which does not differ significantly from the estimated value of 1.95. We note in this figure that the manipulated variable is exhibiting bang-bang control action. This arises because the transfer function of the process has a zero outside the unit circle. To eliminate this behavior, it is necessary to constrain the control action. This is discussed in Harris et al. (1980).

Summary

It has been shown that the best achievable performance, when measured by the mean square error, can be estimated from a time series fit to closed loop process output data alone. It is only necessary to know the integer portion of the process time delay divided by the control interval. No 'identifiability' constraints need be applied. Application of the methodology to pilot plant and production data indicates that this approach gives very credible estimates of the variance that can be realized under minimum variance control.

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Nomenclature

\[ a_i \] = white noise driving force
\[ b \] = number of whole periods of delay, Equation (2)
\[ C(z^{-1}) \] = polynomial in \( z^{-1} \), Equation (13)
\[ d \] = degree of difference in disturbance model
\[ D_t \] = process disturbance at time \( t \)
\[ D_{t+1/b_i} \] = \( b \) step ahead minimum variance forecast
\[ D(z^{-1}) \] = polynomial in \( z^{-1} \), Equation (14)
\[ e_{t+1/b_i} \] = \( b \) step ahead forecast error
\[ f \] = number of integer periods of delay, Equation (2)
\[ F(z^{-1}) \] = polynomial in \( z^{-1} \), Equation (15)
\[ G(z^{-1}) \] = polynomial in \( z^{-1} \), Equation (15)
\[ G_c(z^{-1}) \] = transfer function of controller, Equation (11)
\[ p \] = order of \( \phi(z^{-1}) \)
\[ q \] = order of \( \theta(z^{-1}) \)
\[ r \] = order of \( \delta(z^{-1}) \)
\[ s \] = order of \( w(z^{-1}) \)
\[ T \] = control interval
\[ T(z^{-1}) \] = polynomial in \( z^{-1} \), Equations (6) and (17)
\[ U_t \] = manipulated variable at time \( t \), (deviation variable)
\[ Y_t \] = controlled variable at time \( t \), (deviation from setpoint)
\[ z^{-1} \] = backward shift operator

Greek Symbols

\[ \psi \] = difference operator
\[ \delta(z^{-1}) \] = denominator polynomial in process dynamics,
Equation (1)
\[ \sigma_a \] = standard deviation of \( a_i \)
\[ \sigma_{f+1} \] = standard deviation of \( f+1 \) step ahead prediction error
\[ \tau_d \] = process deadline
\[ \phi(z^{-1}) \] = autoregressive component in disturbance model,
Equation (3)
\[ \theta(z^{-1}) \] = moving average component in disturbance
model, Equation (3)
\[ w(z^{-1}) \] = numerator polynomial in process dynamics,
Equation (1)

Superscripts

\[ - \] = maximum likelihood estimate

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