Performance Assessment Measures for Univariate Feedback Control

LANE DESBOROUGH* and THOMAS HARRIS*

Department of Chemical Engineering, Queen's University, Kingston, Ontario K7L 3N6

A normalized performance index, η(b), is introduced to characterize the performance of feedback control schemes. η(b) is attractive because it provides a measure of the proximity of control to minimum variance control, which is the optimal feedback control provided that the process can be described by a linear transfer function with additive disturbance. Both time domain and spectral interpretations of this performance index are discussed. A fast, simple, on-line method for estimating η(b) is given, along with some of the statistical properties of the estimator. Simulation and industrial examples demonstrate the utility of η(b).

On présente un indice de performance normalisé, η(b) afin de caractériser la performance de schémas de contrôle de rétention. Cet indice η(b) est intéressant car il fournit une mesure de la proximité du contrôle par rapport au contrôle de variance minimum, qui constitue le contrôle de rétention optimal dans la mesure où ce procédé peut être décrit par une fonction de transfert linéaire avec une perturbation supplémentaire. On examine tant le domaine de temps que les interprétations spectrales de cet indice de performance. On présente une méthode rapide, simple et en ligne pour l’estimation de η(b) ainsi que certaines des propriétés statistiques de l’estimateur. Des cas de simulation et des exemples industriels démontrent l’utilité de η(b).

Keywords: control performance assessment, feedback control, time series methods, $R^2$.

Modern manufacturing facilities have hundreds and possibly thousands of automatic control loops. It is nearly impossible to monitor the performance of more than a few of the most critical control loops without some formalized assessment tools. When a poorly performing control loop is identified, it is necessary to diagnose the underlying cause of this poor behavior. For example, poor performance could be due to improper tuning, improper controller structure, changing process dynamics, or an excursion in the process disturbance. In order to be able to improve performance, it is necessary to correctly diagnose the underlying problem. The emerging area of performance assessment provides a means of diagnosing control loop performance using time series and digital signal processing techniques.

Åström (1991) and Åström et al. (1992) present means for assessing achievable performance using PID control. Achievable performance is characterized in terms of bandwidth and dimensionless numbers such as normalized peak error for setpoint and load disturbances and rise time. Shinskey (1990) introduced an Absolute Performance Index (API), which assesses the performance of PID-type controllers (including Smith Predictors) in terms of the integrated absolute error (IAE). These approaches require Laplace transform models for the process and disturbances. The emphasis in these papers is not directed towards using routine operating data to assess controller performance. Instead, the focus is on the use of these measures for selecting the structure and tuning parameters of PID-type controllers at the design stage. Several of these measures can be used online in an expert system environment (Åström et al., 1992). The methods are applicable to processes with noise-free measurements and nonstochastic disturbances, i.e., ramps, steps and exponential rises to new levels.

Harris (1989) showed that a lower bound on the closed loop output variance could be obtained by analyzing routine closed loop operating data. This procedure consists of fitting an autoregressive moving average (ARMA) time series model to the closed loop output data. The lower bound on the performance is calculated by solving a system of linear equations (the Diophantine equation). It is necessary to know the process deadtime in order to implement this procedure. Stanfelj et al. (1991) extended this technique to feedforward/feedback loops. They developed a cross-correlation test to determine when a feedforward control loop is performing optimally in the mean square error sense. As well, they developed an analysis technique to identify whether the feedforward or feedback component of the control loop was responsible for sub-optimal performance. Both of these tests use routine operating data, thus it is not necessary to perturb the process to obtain the necessary information for analysis.

In this paper, we have extended the ideas developed in Harris (1989). A normalized performance index for assessment of controller performance against a benchmark of minimum variance control is introduced. This normalized performance index is a scalar between zero and one, with zero indicating that the process is operating under minimum variance control, i.e., at its optimal performance bound. We show that the normalized performance index can be estimated by linear regression methods and outline two algorithms for its calculation. The normalized performance index may be estimated online using recursive least squares, enabling the use of control charts to monitor changes in performance.

The normalized performance index is then shown to be equivalent to the squared correlation coefficient, $R^2$, used in regression analysis. The statistical properties of the normalized performance index are derived. This enables one to examine the explicit dependence that sample length, process delay, and the autocorrelation structure of the process have on the uncertainty associated with this statistic. Finally, the normalized performance index is shown to have a spectral interpretation. The paper concludes with an application of the proposed methods to a large industrial data set.

Minimum variance control: A performance benchmark

The performance of an existing control loop is often measured against some kind of benchmark. There are many

*Author to whom correspondence may be addressed.
†Current address: Novacor Chemicals Ltd., P.O. Box 5006, Red Deer, AB, T4N 6A1.

1186 THE CANADIAN JOURNAL OF CHEMICAL ENGINEERING, VOLUME 70, DECEMBER, 1992
different measures of control performance, such as offset from setpoint, overshoot, rise-time, and variance. For regulatory control, the latter is an important performance measure since many process and quality release criteria are based on variance. The performance of a control loop might be deemed unacceptable if the variance of the control variable, or output, exceeds some critical value. This criterion, however, fails to recognize the difference between acceptable performance and good control. As discussed in Harris (1989), when the controller is already giving the best possible performance, i.e. minimum variance control, it is not possible to reduce the variability of a process variable by simple re-tuning of the controller or by choosing a more sophisticated linear feedback control algorithm. Although the resulting variability may be unacceptable from a production or sales perspective, the performance of the control loop from a control perspective is good. In these cases, reductions in variability are only achieved by modifying the system. A logical choice for a performance benchmark is thus minimum variance control.

In many instances, the current controller will not be giving minimum variance performance. This may be due to either poor tuning, changes in the process dynamics and disturbances, or deliberate de-tuning. It may not be desirable to implement a minimum variance controller due to the excessive control action that is required when the sampling interval is short relative to the dominant time constant of the process. Also, it may not be possible to implement a minimum variance controller due to non-invertible transfer function elements. Nevertheless, the minimum variance performance gives a lower bound on the variance of the control system, hence it serves as a useful benchmark for assessing control performance. Whether or not the minimum variance is achievable depends on the invertibility of the process transfer function. When the process transfer function is non-invertible, a modified minimum variance controller can be designed which provides an achievable bound on the variance (Åström and Wittenmark, 1984; Peterka, 1972).

A compact test for minimum variance control is to calculate and plot the sample autocorrelation function of $y$, the closed loop deviation from setpoint of the process output. Under the assumption that the controller is minimum variance, the true autocorrelations are zero for lags greater than or equal to $b$, where $b$ is the number of whole periods of delay in the process (Box and Jenkins, 1976; Åström, 1970). The number of whole periods of delay is defined as the number of control intervals that elapse between making a change in the manipulated variable and first observing its effect on the process output. By definition, $b \geq 1$. This analysis assumes that the process output, or a suitable transformation such as its logarithm, can be adequately described by a linear transfer function with additive disturbance. The autocorrelation test for minimum variance control is discussed in Harris (1989) and Stanfelj et al. (1991). Further extensions are discussed in Desborough (1992). We note that in the unlikely event that the controller transfer function has unstable poles, the autocorrelations may be zero beyond lag 0, even though the controller is not minimum variance.

If it is ascertained that the process is operating under minimum variance control, then future reduction in the output variance will not be obtained by re-tuning the controller or implementing a more complicated linear feedback controller. Reduction in variability will only be achieved by reducing the deadtime, implementing feedforward control, eliminating disturbances through equipment modifications, or reduction in material variability, i.e. changing the system or control structure. If the autocorrelations are 0 at and beyond lag $b$, it may be possible to reduce the output variability by employing a nonlinear controller. When the process is nonlinear, then it is possible that better control can be achieved using a nonlinear controller. Some preliminary results on detecting nonlinearities in process data are discussed in Fogal (1991).

In the following sections, methods are developed for characterizing controller performance for the case when the process is not operating under minimum variance control. The normalized performance index

Many industrial processes can be adequately modelled by the superposition of a linear plant model and a linear disturbance model:

$$Y_t - \mu = \omega(B) B^b / \delta(B) u_t + D_t \quad \text{(1)}$$

where $Y_t$ is the measured process output, $\mu$ is the mean of $Y_t$, and $u_t$ is the deviation of the manipulated variable from a reference value required to keep the process at its mean value. $\omega(B)$ and $\delta(B)$ are polynomials in the backward shift operator $B$ ($B Y_t = Y_{t-1}$). $b$ is the number of whole periods of delay in the process. The disturbance $D_t$ represents the effect of all unmeasured disturbances acting on the process output. Frequently $D_t$ can be modelled as the output of an autoregressive integrated moving average (ARIMA) time series of the form

$$D_t = \theta(B) / \phi(B) v^d a_t \quad \text{(2)}$$

where $\{a_t\}$ is a sequence of independently and identically distributed random variables. $\theta(B)$ and $\phi(B)$ are stable polynomials in the backward shift operator $B$ and $v$ is an abbreviation for $1-B$.

Denote the transfer function of the linear time invariant controller being used to regulate $Y_t$ about a fixed setpoint $Y_{sp}$ by $G_c(B)$, i.e.

$$u_t = -G_c (B) (Y_t - Y_{sp}) \quad \text{(3)}$$

It can be readily shown (Harris, 1989) that the closed loop system is described by

$$Y_t - \mu_Y = \psi(B) a_t \quad \text{(4)}$$

where $\mu_Y$ is the mean of $Y_t$ under feedback control. Ideally, $\mu_Y = Y_{sp}$, however this may not always be the case. The monic polynomial $\psi(B)$ can be broken into two parts

$$Y_t - \mu_Y = \psi_1 (B) a_t + \psi_2 (B) a_{t-b} \quad \text{(5)}$$

where $\psi_1 (B)$ is a monic polynomial of order $b-1$:

$$\psi_1 (B) = 1 + \psi_1 B + \ldots + \psi_{b-1} B^{b-1} \quad \text{(6)}$$

The first and second terms in Equation (5) can be interpreted as the $b$-step ahead forecast error and $b$-step ahead forecast, respectively:

$$Y_t - \mu_Y = e_t + \hat{y}_t \quad \text{(7)}$$
The coefficients of $\psi_1(B)$ are obtained by solving a Diophantine equation or long division of the polynomial $\phi(B) \div h(B)$. The $\psi_1(B)$ polynomial is not a function of $\omega(B)$, $b(B)$, or the controller $G_c(B)$. It depends only on the delay in the process and the disturbance model. $\psi_2(B)$ is a function of the process and disturbance models and the controller.

Define the deviation of the measured process output from setpoint as

$$y_i = Y_i - Y_{sp} \quad (8)$$

The variance of $y_i$ is given by

$$\text{var}\{y_i\} = \text{var}\{\psi_1(B)a_i\} + \text{var}\{\psi_2(B)a_{i-b}\} + 2\text{cov}\{\psi_1(B)a_i, \psi_2(B)a_{i-b}\}$$
$$= \text{var}\{e_i\} + \text{var}\{\hat{y}_i\} + 2\text{cov}\{e_i, \hat{y}_i\} \quad (9)$$

The covariance term vanishes due to the independence of the $\{a_i\}$’s.

When a minimum variance controller is implemented, the $b$-step ahead forecast $\hat{y}_i$ is set to zero (Åström, 1970; Harris, 1989), hence the second term on the right hand side of Equation (9) also vanishes, leaving

$$\text{var}\{y_i\} = \text{var}\{\psi_1(B)a_i\} = \text{var}\{e_i\} = \sigma_{mv}^2 \quad (10)$$

The controller which minimizes the variance also minimizes a broad class of symmetric and nonsymmetric cost functions, (Harris, 1992).

In the more general case where the controller is not minimum variance,

$$\sigma_y^2 = \sigma_{mv}^2 + \sigma_{\hat{y}}^2 \quad (11)$$

where $\sigma_{\hat{y}}^2$ is the variance of $\hat{y}_i$. Thus, any controller which is not minimum variance must inflate the variance of $y_i$ at the sampling intervals. We find it more useful to assess control not in terms of the variance of $y_i$ but in terms of the mean square error of $y_i$:

$$\text{mse}(y_i) = \sigma_y^2 + \mu_y^2 \quad (12)$$

where $\mu_y^2$ is the mean deviation from setpoint.

Now that the theoretical minimum variance is available, it is possible to express the current controller’s performance in terms of the theoretical minimum variance. There are many ways in which the existing controller’s performance can be compared with the theoretical minimum variance — here we suggest only two. First, one could define a performance index such as

$$\xi(b) = \text{mse}(y_i) / \sigma_{mv}^2 = [\sigma_y^2 + \mu_y^2] / \sigma_{mv}^2 \quad (13)$$

$\xi(b)$ is always greater than one but has no upper bound. In addition to having no upper bound, it will be shown later that $\xi(b)$ has some undesirable sampling properties. We have found it more convenient to define a normalized performance index, $\eta(b)$, which expresses the fractional increase in output mean square error that arises from not implementing a minimum variance controller. Define

$$\eta(b) = 1 - \sigma_{mv}^2 / \text{mse}(y_i) = 1 - \sigma_y^2 / [\sigma_y^2 + \mu_y^2] \quad (14)$$

$\eta(b)$ is bounded by [0, 1].

When the process mean is at setpoint, $\eta(b)$ is not dependent on the variance of $\{a_i\}$. We will later show that an advantage of the normalized performance index $\eta(b)$ which we have just introduced is that it is directly analogous to the multiple coefficient of determination $R^2$ found in multiple linear regression. In fact, later on we will use regression methods to arrive at an estimate for $\eta(b)$.

There is, however, an important difference in the way that $\eta(b)$ and $R^2$ are interpreted. In regression analysis, the intent is to have the majority of the variance of the dependent variable explained by the model, i.e. a value of $R^2$ close to one. In contrast, for this analysis we desire that the predictable component in $y_i$ be very small, i.e. a value of $\eta(b)$ close to zero.

In many cases, the setpoint will not be fixed. We denote the changing setpoint by $Y_{sp,t}$. The optimal minimum variance controller is achieved by implementing a two degree of freedom controller (Harris and MacGregor, 1987):

$$u_t = -G_{c,1} (B) Y_t + G_{c,2} (B) Y_{sp,t} \quad (15)$$

The minimum variance controller for this structure can be readily determined once a model for the setpoint changes is specified. However, in most applications, a single degree of freedom controller, i.e. Equation (3), is usually employed. As well, the setpoints are not scheduled in advance, i.e. they are randomly occurring. In this case, we presume that the value of the setpoint at time $t + b$ equals the current setpoint. For this situation, the closed loop behaviour is described by

$$y_i - \mu_y = \psi'(B)a_i \quad (16)$$

where $\psi'(B)$ is a polynomial in the backward shift operator and $a_i'$ is a sequence of independent and identically distributed random variates. The $\psi'(B)$ polynomial is obtained from the model for the stochastic disturbance and the model for the randomly occurring setpoint changes through a process known as spectral factorization (Harris and MacGregor, 1987). The details of this are beyond the scope of this paper. The important result is that when a single degree of freedom controller is employed and setpoint changes are not scheduled, the performance measures described in this section are valid.

ESTIMATION OF THE NORMALIZED PERFORMANCE INDEX

To obtain an estimate for $\eta(b)$, it is necessary to estimate $\sigma_{mv}^2$. It is shown in Harris (1989), that an estimate of $\sigma_{mv}^2$ denoted by $\hat{\sigma}_{mv}^2$ can be obtained from routine operating data. The procedure consists of fitting an autoregressive moving average (ARMA) time series of the form

$$y_i - \mu_y = F(B) / G(B)a_i \quad (17)$$

The population mean $\mu_y$ is usually replaced by its sample value, $\bar{y}$ (Box and Jenkins, 1976). The structure and parameters of the polynomials $F(B)$ and $G(B)$ are found using any standard time series package. The $\psi(B)$ polynomial is extracted from the $F(B)$ and $G(B)$ polynomials by solving a Diophantine equation or long dividing the denominator into the numerator. It is only necessary that the process delay be known and the controller $G_c(B)$ be stable to implement this procedure.
It is important to keep in mind that the actual model used to fit the closed loop output data is not important, so long as it adequately fits the data and we can estimate the b-step ahead forecast error. Therefore we are not limited to the moving average (MA) form in Equation (5) or the autoregressive moving average (ARMA) form in Equation (17). Indeed, we can take advantage of the properties of autoregressive (AR) models and fit a model of this form to the data.

In the next sections, we will outline two methods for estimating the normalized performance index.

Linear regression approach

In the previous section, we introduced the normalized performance index, \( \eta(b) \). In this section we show a simple way to estimate \( \eta(b) \) from routine closed loop process data using linear regression methods. This approach eliminates the necessity of solving a Diophantine equation or performing polynomial long division.

The process output under feedback control is given by Equation (4). By first rearranging Equation (4) and evaluating at \( y_{t-b} \), and then substituting the result into Equation (5), we see that

\[
y_t - \mu_y = e_t + \psi_2(B)/\psi(B) [y_{t-b} - \mu_y] \tag{18}
\]

Since the closed loop system is presumed stable, \( \psi_2(B)/\psi(B) \) must form a convergent series in the backward shift operator \( B \). Consequently, the closed loop system can be represented by an expression of the form

\[
y_t - \mu_y = e_t + \sum_{k=1}^{\infty} \alpha_k (y_{t-b-k+1} - \mu_y) \tag{19}
\]

In practice, the infinite series is truncated after \( m \) terms. Thus the closed loop behaviour can be expressed as

\[
y_t - \mu_y = e_t + \sum_{k=1}^{m} \alpha_k (y_{t-b-k+1} - \mu_y) \tag{20}
\]

To estimate the autoregressive parameters \( \{\alpha_k\} \) using a sample of closed loop data \( \{y_1, \ldots, y_n\} \), one can fit a lagged regression using Equation (20), which in matrix notation is

\[
\tilde{y} = \tilde{X} \lambda + \tilde{\varepsilon} \tag{21}
\]

where

\[
\begin{bmatrix}
\tilde{y}_n \\
\tilde{y}_{n-1} \\
\vdots \\
\tilde{y}_{b+m}
\end{bmatrix}
= \begin{bmatrix}
\tilde{y}_{n-b} & \tilde{y}_{n-b-1} & \cdots & \tilde{y}_{n-b-m+1} \\
\tilde{y}_{n-b} & \tilde{y}_{n-b-2} & \cdots & \tilde{y}_{n-b-m} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{y}_{n-b} & \tilde{y}_{n-b-2} & \cdots & \tilde{y}_{n-b-m}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_m
\end{bmatrix}
\]

\[
\tilde{\varepsilon} = \begin{bmatrix}
e_t \\
\vdots \\
e_{t-b}
\end{bmatrix}
\tag{22}
\]

where \( n \) is the sample length and \( \bar{y}_n = y_n - \mu_y \) is the mean corrected deviation from setpoint. The parameter estimates are found by solving the set of linear equations

\[
(\tilde{X}^T \tilde{X}) \hat{\lambda} = \tilde{X}^T \tilde{\varepsilon} \tag{23}
\]

\( \mu_y \) is usually replaced by its sample value \( \bar{y} \). The residual mean square error is given by

\[
s_e^2 = (\bar{y} - \hat{X} \hat{\lambda})^T (\bar{y} - \hat{X} \hat{\lambda}) / (n - b - 2m + 1) \tag{24}
\]

which provides an estimate for \( \sigma_{mv}^2 \).

The least squares estimate for the normalized performance index, \( \eta(b) \), is

\[
\hat{\eta}(b) = 1 - \frac{n - b - m + 1}{n - b - 2m + 1} \frac{(\bar{y} - \hat{X} \hat{\lambda})^T (\bar{y} - \hat{X} \hat{\lambda})}{\bar{y}^2 + \bar{y}^2} \tag{25}
\]

This can be recognized as the adjusted multiple coefficient of determination, or adjusted \( R^2 \) which is used in regression analysis (Mason et al., 1990). In most time series applications, \( n \gg m \), hence \( \hat{\eta}(b) \) is the same as \( R^2 \) when \( \bar{y} = 0 \), i.e. the process averages its setpoint.

Recursive least squares approach

While the normalized performance index is useful in its own right, it can be especially valuable to calculate it sequentially in time to detect change points. The previous method for estimating the normalized performance index is not directly suited to online implementation. If one wants to estimate the normalized performance index online, it is better to use a recursive least squares algorithm to provide estimates for the parameters in Equation (20). In estimating the parameters \( \hat{\alpha} \), we replaced \( \mu_y \) with its sample estimate \( \bar{y} \).

In a recursive algorithm \( \bar{y} \) is not available. In this case, the model for the closed loop behaviour is modified to include a constant term \( \alpha_0 \). The parameters \( \{\alpha_0, \ldots, \alpha_m\} \) are estimated using Equation (21), with \( \bar{y}_k \) replacing \( y_k \) for all \( k \). The parameter vector, \( \alpha \) is augmented by \( \alpha_0 \), and the \( \tilde{X} \) matrix is augmented by a leading column of ones. From these parameter estimates, an estimate of the minimum variance performance at time \( t \), denoted \( s_{mv,t}^2 \), can be easily estimated, in turn providing an estimate for the normalized performance index.

Recursive least squares methods usually employ a "forgetting factor", \( \lambda \), which enables more emphasis to be placed on recent data. Typically, recursive least squares algorithms minimize a cost function of the form

\[
J = (y - X \alpha)^T \Lambda (y - X \alpha) \tag{26}
\]

where \( \Lambda \) is a diagonal matrix with elements \( \lambda, \lambda^2, \ldots, \lambda^m \). An estimate of the minimum variance performance at time \( t \) is given by

\[
s_{mv,t}^2 = \lambda s_{mv,t-1}^2 + e_t^2 \tag{27}
\]

where \( e_t \) is the prediction error at time \( t \).

The normalized performance index is calculated as

\[
\hat{\eta}(b) = 1 - \frac{s_{mv,t}^2}{s_{y,t}^2} \tag{28}
\]

where \( s_{y,t}^2 \) is the exponentially weighted moving mean square error

\[
s_{y,t}^2 = \lambda s_{y,t-1}^2 + y_t^2 \tag{29}
\]

The use of recursive online estimation of the normalized performance index is not limited to monitoring for change points in controller performance. For instance, if the process is non-linear and the dynamics are slow enough that the process can be considered locally linear, recursive estimation of the normalized performance provides a local estimate of the controller performance. Alternatively, $\hat{\eta}(b)$ can be used online as a tuning tool to show immediately whether the tuning changes have improved or degraded the controller's performance. This assumes that the disturbance model does not change.

**CHOICE OF MODEL ORDER**

In fitting the closed loop output model, the infinite order autoregressive time series is truncated to order $m$. The choice of autoregressive model order has attracted a considerable amount of attention in the literature. Various measures such as the Akaike Information Criteria (AIC) have been used to elucidate the best model order. Stoica et al. (1986) gives a comprehensive review of model structure selection techniques. Typically, we calculate the normalized performance index for increasing values of $m$, starting with $m \approx 5$. The model order $m$ is increased until the normalized performance index estimate shows no appreciable change.

**INTERPRETATION OF THE NORMALIZED PERFORMANCE INDEX**

As already noted, the $\eta(b)$ for which we have just introduced is directly analogous to the multiple coefficient of determination found in multiple linear regression. Hence interpretations of $R^2$ in regression analysis (Rodgers and Nicewander, 1988) are directly applicable to $\eta(b)$. The most useful interpretation of $\eta(b)$ for us is that it is the proportion of the process variability not accounted for by the $b$-step ahead forecast error. Ideally, the deviation $\gamma_i$ will have no predictable component and the process variability will be equal to the $b$-step ahead forecast error variance. Nelson (1976) provides useful insight into the interpretation of $R^2$ for time series models. He notes that $R^2$ provides a measure of the predictability of a time series given its past history. Furthermore, he notes that $R^2$ serves as a scalar measure of autocorrelation in the time series, since it can be written in terms of the autocorrelation function alone. Nelson also makes the important observation that $R^2$ is independent of the noise variance $\sigma^2_y$, hence it provides a measure of performance which is not affected by the magnitude of the disturbances. When there are frequent setpoint changes, $\eta(b)$ can be used as an indicator of the predictable component in the setpoint changes.

The normalized performance index provides a measure of the predictable component in the process output $b$ steps into the future. For a minimum variance benchmark, we are interested in choosing $b$ as the number of periods of process delay. This is not always appropriate. As an example, consider a controller which seeks to minimize the 10-step ahead prediction error (Ydstie et al., 1985). We could use $\hat{\eta}(10)$ as a performance measure for this controller. As another example, if we are using a feedforward variable which effectively reduces the deadtime of the system by three periods, then $\hat{\eta}(b - 3)$ can be used as a performance measure. Finally, $\hat{\eta}(1)$ provides a measure of the performance if the delay is reduced to its minimum value of 1. This particular form of the normalized performance index is discussed by DeVries and Wu (1978). We recognize that most processes have nonlinearities. The performance index is always interpreted as the best achievable control using a linear model and a linear controller.

**DISTRIBUTION AND APPROXIMATE MOMENTS OF $\hat{\eta}(b)$**

It is important to be able to characterize the distribution of $\hat{\eta}(b)$ and those factors such as the delay and autocorrelation structure which affect its mean, variance, and higher moments. Approximate confidence intervals for $\hat{\eta}(b)$ can be calculated once its distributional properties are known. This enables us to construct control charts to monitor changes in the closed loop performance. In Harris (1989), the distribution of $s_{\eta}/\bar{s}_{\eta}$ for a specific example was investigated using Monte-Carlo simulation. In this section we will develop approximate moments for $\hat{\eta}(b)$. The only assumption we make is that the $\{a_i, \sigma_i\}$ are independent, identically distributed random normal deviates.

The distribution of the sample multiple coefficient of determination, $R^2$, was first derived by Fisher (1928). This result cannot be utilized in our case since the distribution was derived under the assumption that the data has no time correlation. The exact distributional form for $R^2$ for the case of serially correlated observations is unresolved. However, we can still gain useful insight into the distribution of $\hat{\eta}(b)$ through estimation of the moments of $\hat{\eta}(b)$.

Hosking (1979, 1991) showed that for autoregressive time series with $b = 1$, $\hat{\eta}(b)$ is asymptotically normally distributed:

$$\hat{\eta}(b) \sim N(\eta(b), (4/n) [1 - \eta(b)]^2 \sum_{i=1}^{\infty} \rho_k^2) \ldots (30)$$

where $N(\mu, \sigma^2)$ denotes the normal distribution with mean $\mu$ and variance $\sigma^2$, and $\rho_k$ is the true autocorrelation at lag $k$. As shown in the Appendix, $\hat{\eta}(b)$ may be calculated as the ratio of two quadratic forms. By approximating this ratio of quadratic forms by a first order Taylor series, the approximate moments for the general $b$-step ahead case can be derived in terms of output and error autocorrelations ($\rho$ and $\rho_c$). The first two moments for the distribution of $\hat{\eta}(b)$ for large $n$ are approximately:

$$\text{mean}[\hat{\eta}(b)] = \eta(b) \ldots (31)$$

$$\text{var}[\hat{\eta}(b)] = (4/n) [1 - \eta(b)]^2$$

$$\left(\sum_{i=1}^{\infty} (\rho_k - \rho_{c,k})^2 + \sum_{i=1}^{\infty} \rho_k^2\right) \ldots (32)$$

Estimates for the third moment are given in Desborough and Harris (1992). Moments for the performance index in Equation (13) are also given in the Appendix, where we note that the variance is proportional to the square of $\hat{\xi}(b)$.

The variance of $\hat{\eta}(b)$ is a complex function of the true normalized performance index, the autocorrelation structure of the process, and the process delay. As the performance deteriorates, $(1 - \eta(b))^2$ decreases. However, as $\eta(b)$ increases, the autocorrelations increase, hence the term inside the large brackets in Equation (32) will increase. Thus for a fixed sample length, the variance of $\hat{\eta}(b)$ is affected by two competing terms.
In the recursive least squares implementation, it is necessary to specify a value for the “forgetting factor” \( \lambda \). If \( \lambda \) is chosen as

\[
\lambda = (n - 1)/(n + 1)
\]

then the variance calculated from both least squares methods is the comparable (Desborough and Harris, 1992).

**Simulation**

In order to evaluate the accuracy of the approximations for the moments of \( \hat{\eta}(b) \), the following simple process was simulated:

\[
y_i = u_{i-b} + (1 - 0.2B)/(1 - B) u_i
\]

(34)

When a simple integral controller \( u_i = -K y_i \) is implemented, the closed loop transfer function is (assuming no setpoint changes):

\[
y_i = (1 - 0.2B)/(1 - B + KB^b) u_i
\]

(35)

The controller tuning parameter \( K \) which minimizes the variance of \( y_i \) was found analytically for process delays of 1 to 5. \( K \) was determined by expressing the variance as an explicit function of \( K \) and determining the stationary points of this expression. A symbolic manipulation package was used to perform these calculations.

To perform the simulation, a series of standard normal random variates were generated using the method of Park and Miller (1988). These random variates were tested for independence using the modified Box-Pierce statistic (Wei, 1990). The series was not used if the hypothesis that all the autocorrelations from lag 1 to lag 10 are zero was rejected at the 95% confidence level. Once an acceptable \( \{a_i\} \) series was generated, the process output was then calculated using Equation (35).

For each \( b = 1\ldots5 \) and each \( n = 100,200,300 \), two thousand realizations of Equation (35) were generated. \( \hat{\eta}(b) \) was estimated using the linear regression method. The sample mean and variance were calculated. The probability density function of \( \hat{\eta}(b) \) was estimated using kernel density estimates, according to the method of Silverman (1986). Kernel estimates provide a much smoother estimate of the probability density function than a histogram. The density estimates are shown in Figure 1.

Some observations can be made about the shapes of the distributions. First, the distributions are narrow for small delays and grow progressively broader as the delay increases. Second, the distributions grow narrower as the number of samples increases. As well, the distribution approaches normality as the number of samples and periods of delay increase.

To study the effects of different controllers on the distribution of \( \hat{\eta}(b) \), the controller was overtuned by increasing \( K \) to 1.5 times the optimal value, \( K_{opt} \). For each \( b = 1\ldots5 \) and each \( n = 100,200,300 \), two thousand realizations of Equation (35) were generated. The sample mean and variance were calculated. Again, some comments about the density estimates are in order. As before, the distributions are narrow for smaller delay and larger number of samples, and broad for larger delay and smaller number of samples. The distributions are much broader than the corresponding optimally tuned distributions.

The sample variances for \( \hat{\eta}(b) \) are plotted in Figures 2 and 3 for \( K = K_{opt} \) and \( K = 1.5K_{opt} \), respectively. Also shown on these plots are the theoretical variances calculated from
Equation (32). The agreement is excellent between the theoretical values and the simulation values. Both show an increase in the variance with increasing δ, and decreasing sample size.

**Choice of sample length**

The choice of sample is subject to three considerations. First, with a short sample, the statistics are more susceptible to outliers. Second, the distribution for \( \hat{\eta}(b) \) can be very non-normal for short sample lengths. Third, with too long a sample, the underlying process may change and we begin to simply average the data. In these instances we do not have an appropriate representative sample to judge the performance of the control loop.

The choice of sample length can be regarded as a trade-off between narrow confidence limits for estimates and loss of information about current control performance. Once \( \eta(b) \) has been estimated, an approximate confidence interval can be constructed using the variance of \( \hat{\eta}(b) \) by replacing the population parameters in Equation (32) with their sample values. If it is found that the confidence interval for \( \hat{\eta}(b) \) is so wide as to make inference about \( \eta(b) \) meaningless, then an increase in sample size is required. However, it may happen that the required sample size is so large that one doubts the value of the analysis. Thus the distribution of \( \hat{\eta}(b) \) provides valuable insight as to whether the sample size is appropriate.

**SPECTRAL INTERPRETATION OF \( \eta(b) \)**

The normalized performance index is a scalar characterization of the closed loop behaviour which can be regarded as an extreme form of data compression. The normalized performance index provides no insight into why the performance differs from minimum variance. A more detailed analysis of performance is obtained by examining the spectral interpretation of \( \eta(b) \).

The spectral analysis of time series is discussed in many textbooks, (Jenkins and Watts, 1968; Wei, 1990; Priestley, 1981). The closed loop process output is given by Equation (4):

\[
y(t) = \psi(B) a(t) \tag{4}
\]

The spectrum of \( y(t) \), denoted by \( h_y(f) \), is

\[
h_y(f) = \sigma_y^2 |\psi(e^{-2\pi jf})|^2 \tag{36}
\]

where \( 0 \leq f \leq 0.5 \) and \( j = \sqrt{-1} \). \( f \) is the frequency, in cycles per sample. \(| \cdot | \) denotes the magnitude of the function \(( \cdot )\). The variance of \( y(t) \) is the integral of the spectrum:

\[
\sigma_y^2 = \int_0^{0.5} |\psi(e^{-2\pi jf})|^2 df \tag{37}
\]

The integral of the spectrum between two frequencies is the variance attributed to that frequency range. The spectrum thus provides a useful tool for analyzing process signals by decomposing the variance into its frequency components. For example, a large spectral peak in the low frequencies could indicate an under-tuned controller. Conversely, a large spectral peak in the high frequencies could indicate an over-tuned controller. The use of the spectrum for diagnosing control loop performance is discussed in Pryor (1982), DeVries and Wu (1978) and Ohtsu and Kitagawa (1984).

The output spectrum by itself provides no information as to the achievable reduction of variance in any frequency range. To gain information about the achievable reduction in variance, it is necessary to determine the spectrum of the process under minimum variance control. The spectrum of the process under minimum variance control is given by

\[
h_{mv}(f) = \sigma_y^2 |\psi(e^{-2\pi jf})|^2 \tag{38}
\]

A plot of \( h_y(f) \) and \( h_{mv}(f) \) provides information on how the spectral content of \( y(t) \) must be modified if minimum variance control is to be achieved.

To illustrate this, consider the simple example given by Equation (35), when \( K = 0.5K_{opt} \), \( K = K_{opt} \), and \( K = 1.5K_{opt} \). The spectra for the three cases are given in Figure 4, along with the spectra for the theoretical minimum variance controller. For the undertuned case \( K = 0.5K_{opt} \), we see that the majority of the variance appears in the low frequency range, confirming that the process tends to wander from setpoint for prolonged periods of time. For the over-tuned case \( K = 1.5K_{opt} \), we see that the majority of the variance appears as a peak in the mid-frequency range, confirming that the controller tends to make the process cycle excessively. The optimally tuned controller most closely resembles the spectrum of the minimum variance controller.

The spectrum of the output is useful in its own right, since it indicates the frequencies at which the majority of the variance enters the process. As already shown, however, it does not indicate the frequencies in which the performance differs from minimum variance control. A spectral interpretation of the normalized performance index can be defined as follows. Denote the difference between \( h_y(f) \) and \( h_{mv}(f) \) by \( \Delta(f) \), i.e.

\[
\Delta(f) = h_y(f) - h_{mv}(f) \tag{39}
\]

Integrating this equation between 0 and 0.5 and dividing by \( \sigma_y^2 \), one obtains

\[
\eta(b) = \int_0^{0.5} \frac{\Delta(f)}{\sigma_y^2} df = \int_0^{0.5} \frac{[h_y(f) - h_{mv}(f)]}{\sigma_y^2} df \tag{40}
\]
Thus the normalized performance index is related to the integral of the difference between the two spectra $h_s(f)$ and $h_{mv}(f)$.

An alternative interpretation of $\eta (b)$ is obtained from the spectral interpretation of Equation (7) (Priestley, 1981):

$$h_s(f) = h_3(f) + h_{mv}(f) + h_{se}(f) + h^*_{se}(f) \ldots \ldots \ldots (41)$$

where

$$h_3(f) = |\psi_2(e^{-2\pi i f})/\psi(e^{-2\pi i f})|^2 h_s(f) \ldots \ldots \ldots (42)$$

and

$$h_{se}(f) = \psi_1(e^{-2\pi i f}) \psi_2(e^{-2\pi i f}) e^{-2\pi i f b} a_n^2 \ldots \ldots \ldots (43)$$

and $h^*_{se}(f)$ is the complex conjugate of $h_{se}(f)$. The latter quantity is the cross spectrum of $\psi_1(B) a_i$ and $\psi_2(B) a_{t-b}$. Integrating Equation (41) and dividing by $\sigma^2$ one obtains

$$\eta (b) = \int_0^{0.5} [h_s(f)/\sigma^2] df + 2(\sigma^2 / \sigma^2) \text{cov} \{ e, y \} \ldots \ldots \ldots (44)$$

where $\text{cov} \{ e, y \}$ is the covariance between $\psi_1(B) a_i$ and $\psi_2(B) a_{t-b}$. As shown in Equation (9), this value is 0. We note that the integrand in Equation (44) is the normalized spectrum of the predictor, $h_s(f)/\sigma^2$. Equation (44) shows that $\eta(b)$ as defined in Equation (40) is indeed positive.

It is therefore convenient to define the spectral interpretation of the normalized performance index by

$$\eta(b, f) = (h_s(f) - h_{mv}(f))/\sigma^2 \ldots \ldots \ldots \ldots (45)$$

If the controller is minimum variance, then $\eta(b, f)$ is zero everywhere. When the controller is not minimum variance, $\eta(b, f)$ indicates the frequency range in which the controller's performance differs from minimum variance. Note that $\eta(b, f)$ is not a spectrum since it can take on negative values.

**Estimation of $\eta (b, f)$**

The spectrum of $\gamma_i$ can be estimated using nonparametric techniques, such as windowing the autocorrelation function (Jenkins and Watts, 1968), or parametric techniques such as fitting autoregressive models to the data. We will use the latter method. In this approach, the closed loop is represented by an autoregressive model of the form

$$\gamma_i = \alpha_1 \gamma_{i-1} + \ldots + \alpha_m \gamma_{i-m} + e_i \ldots \ldots \ldots (46)$$

The parameters $[\alpha_1 \ldots \alpha_m]$ are estimated using the techniques discussed in earlier parts of the paper. The spectrum of the process is given by

$$h_s(f) = s^2 / [1 - \alpha_1 e^{-2\pi i f} - \ldots - \alpha_m e^{-2\pi imf}]^2 \ldots \ldots \ldots (47)$$

To estimate the spectrum of the minimum variance controller, $h_{mv}(f)$, we must estimate the parameters $[\hat{\psi}_1 \ldots \hat{\psi}_{b-1}]$ in Equation (6), using the following recurrence relation:

$$\hat{\psi}_i = \sum_{k=i-1}^{i-k} \alpha_k \hat{\psi}_{i-k} \quad i = 1 \ldots b - 1 \ldots \ldots \ldots (48)$$

where $\hat{\psi}_0 = 1$.

**Performance assessment industrial example**

In order to illustrate the above performance measures, an industrial data set was measured from a chemical process. The data consist of process outputs from two controllers in a cascade structure. The top level in the cascade outputs a setpoint to the bottom level of the cascade, which in turn provides a setpoint to a direct digital controller. Both controllers are proportional-integral-derivative (PID) controllers. The cascade structure is shown in Figure 5. There are 13800 equispaced samples. The outputs have been standardized to have mean zero and variance one. The delay for the top level is 17 and the delay for the bottom level is 3. The first 1000 data points for the top and bottom levels are shown in Figures 6 and 7, respectively.

As a first test for minimum variance control, the sample autocorrelations were calculated using a sample consisting of the first 1000 data points. The sample autocorrelations for the top and bottom levels, along with their respective 95% confidence limits, are shown in Figures 8 and 9. Clearly, neither controller is operating near minimum variance.
NORMALIZED PERFORMANCE INDEX

Recursive least squares was used to estimate the normalized performance index for both the top and bottom level. In both cases, the model order was chosen to be 10. The "forgetting factor" was set to $\lambda = 0.998$, giving an effective sample length of 1000 (see Equation (33)).

In addition to the recursive least squares estimates for the normalized performance index, ordinary least squares estimates were calculated every 1000 samples for both levels, with $n = 1000$. Approximate 95% confidence intervals for $\hat{\eta}(b)$ were estimated using Equation (32), with the sample values substituted for the population parameters.

A plot of the recursive and ordinary least squares estimates of the normalized performance index for the top level in the cascade is shown in Figure 10. Clearly, the performance is not close to minimum variance. The recursive and ordinary least squares estimates are in excellent agreement with each other. The normalized performance index indicates that a change occurred at approximately observation 1000 which caused the normalized performance index to increase. Potential causes for this change were not disclosed. Possible causes for this sudden change in the normalized performance index include changes in the form of the process disturbances, changes in controller tuning, or changes in the process dynamics due to such things as a rate change, product change, or process upset. Sudden changes in the normalized performance index cannot be attributed to changes in the magnitude of the process disturbance, since $\eta(b)$ is immune to such effects.

A plot of the recursive and ordinary least squares estimates of the normalized performance index for the bottom level in the cascade is shown in Figure 11. In contrast to the top level, the performance is very close to minimum variance. Again, the agreement between recursive and ordinary least squares estimates is very good. The principle reason for the cyclic behaviour of the normalized performance index for the bottom level was that the sample mean deviated from the process setpoint by widely varying amounts.

One thing which is very clear from the above analysis is that for an improvement in control to be realized, the loop to concentrate on is the top level, since the bottom level is already operating very close to minimum variance control. There will be no appreciable performance benefits derived from re-tuning the bottom level, since it is already operating very close to minimum variance.

SPECTRAL INTERPRETATION

Using the first 1000 observations, we computed the spectrum of the output and theoretical minimum variance
controller for both the top and bottom levels. It has already been shown that the top level controller is not giving minimum variance control. To examine why control is poor and how it can be improved, one can examine the data itself and the autocorrelation function, Figures 6 and 8, respectively. These both indicate that the controller is probably under-tuned since the closed-loop behaviour is so sluggish. The spectrum of the output, Figure 12, and the spectrum of the theoretical minimum variance controller, Figure 13, indicate that the majority of the variance in the top level output is due to a low frequency component, again indicating that the top level is under-tuned. Furthermore, the spectra in Figures 12 and 13 indicate that there is a tremendous discrepancy between the existing control and the theoretical minimum variance control, especially in the very low frequency range. With better tuning or the implementation of a better feedback controller, the variance in the top level output could be reduced potentially by a factor of 10.

The spectrum of the output for the bottom level is shown in Figure 14, along with the theoretical minimum variance spectrum. The two are very similar, indicating that the existing control is very close to minimum variance. As with the top level, most of the difference in the spectra occurs in the low frequency range, indicating that the bottom level control is slightly undertuned.

SUMMARY

An analysis of the performance of the industrial example revealed that the top level is undertuned and is not operating anywhere near minimum variance, and that the bottom level is operating very close to minimum variance. Any improvement in the overall performance of the control scheme will come from the top loop, since the bottom loop is already operating very close to minimum variance control.

Future considerations

The autocorrelation test for minimum variance control for the case of an unstable controller remains to be resolved. A logical extension of this work is to study the effects of feedforward variables. Initial research in this area is reported in Desborough and Harris (1992) and Desborough (1992). When a feedforward variable is also included in the lagged regression model, the benefits derived from the inclusion of feedforward control can be measured, even if this variable is not included in the present control scheme. It is possible to use this as a method for selecting appropriate feedforward variables.

Conclusions

A method of assessing the performance of feedback controllers was developed. It was shown that by using routine closed loop process output data alone, an existing controller's performance could be measured against a benchmark of minimum variance control using the normalized performance index η(b). Two methods of estimating the normalized performance index were given. Approximate moments for η(b) were derived, and a spectral interpretation of η(b) was presented. Finally, an industrial example was used to demonstrate the effectiveness of the performance assessment tools described above.

APPENDIX

Moments of η(b)

To derive the approximate moments of η(b), first note that

\[ η(b) = 1 - \frac{s_1}{s_2} \]  \hspace{1cm} (A.1)

where \( s_1 \) and \( s_2 \) represent the sample variances. This can also be written as a ratio of two quadratic forms by letting \( U = e^T e \) and \( V = y^T y \), so that

\[ η(b) = 1 - \frac{U}{V} \]  \hspace{1cm} (A.2)

where \( e \) and \( y \) are the residual vector and output vector, respectively:

\[
e = \begin{bmatrix} e_n \\ e_{n-1} \\ \vdots \\ e_1 \end{bmatrix}, \quad y = \begin{bmatrix} y_n \\ y_{n-1} \\ \vdots \\ y_1 \end{bmatrix}
\]  \hspace{1cm} (A.3)

We can then approximate \( η(b) \) by a first order Taylor series approximation

\[ η(b) \approx 1 - (U_0/V_0) - (1/V_0) \left( U - U_0 \right) + (U_0/V_0^2) \left( V - V_0 \right) \]  \hspace{1cm} (A.4)

where

\[ U_0 = E[U] = n \sigma_u^2 \]  \hspace{1cm} (A.5)

\[ V_0 = E[V] = n \sigma_v^2 \]
Clearly, the approximate first moment is

\[ \text{mean}[\hat{\eta}(b)] = E[\hat{\eta}(b)] = E[1 - U/V] = 1 - U_0 / V_0 \]
\[ = 1 - \sigma_u^2 / \sigma_v^2 = \eta(b) \]  
(A.6)

To calculate the variance, take the expectation of the square of the Taylor series approximation of \( \eta(b) \)

\[ \text{var}[\hat{\eta}(b)] = E[(U/V - (U_0 / V_0)^2)] = (1/V_0^2) \text{var}(U) \]
\[ + (U_0^2 / V_0^2) \text{var}(V) - 2(U_0 / V_0) \text{cov}(U, V) \]  
(A.7)

Using the method of Kendall and Stuart (1958), the variances in Equation (A.7) may be expressed in terms of traces of products of the covariance matrices

\[ \text{var}(U) = 2 \text{trace}(\Sigma_U \Sigma_U) \]
\[ \text{var}(V) = 2 \text{trace}(\Sigma_U \Sigma_V) \]
\[ \text{cov}(U, V) = 2 \text{trace}(\Sigma_U \Sigma_V) \]  
(A.8)

where
\[ \Sigma_U = \text{Toeplitz}[\gamma_0, \gamma_1, \ldots, \gamma_{n-1}] \]
\[ \Sigma_E = \text{Toeplitz}[\gamma_{n0}, \gamma_{n1}, \ldots, \gamma_{n,n}, 0, \ldots, 0] \]  
(A.9)

\( \gamma_k \) and \( \gamma_{n,k} \) represent the autocovariance and residual autocovariance, respectively, for lag \( k \).

From the above results follow:

\[ \text{var}(U) = 2 \sigma_u^2 \left(n + 2 \sum_{k=1}^{n-1} (n - k) \rho_{u,k}^2 \right) \]
\[ \text{var}(V) = 2 \sigma_v^2 \left(n + 2 \sum_{k=1}^{n} (n - k) \rho_{v,k}^2 \right) \]  
(A.10)

\[ \text{cov}(U, V) = 2 \sigma_u^2 \sigma_v^2 \left(n + 2 \sum_{k=1}^{n-1} (n - k) \rho_{u,k} \rho_{v,k} \right) \]

If \( n \) is large and the autocorrelations decay relatively quickly, then the above asymptotically tend to

\[ \text{var}(U) = 2n \sigma_u^2 \left(1 + 2 \sum_{k=1}^{n-1} \rho_{u,k}^2 \right) \]
\[ \text{var}(V) = 2n \sigma_v^2 \left(1 + 2 \sum_{k=1}^{n} \rho_{v,k}^2 \right) \]  
(A.11)

\[ \text{cov}(U, V) = 2n \sigma_u^2 \sigma_v^2 \left(1 + 2 \sum_{k=1}^{n-1} \rho_{u,k} \rho_{v,k} \right) \]

hence the approximate second moment is

\[ \text{var}[\hat{\eta}(b)] \approx (4/n) \left[1 - \eta(b)\right]^2 \left(\sum_{k=1}^{n-1} (\rho_k - \rho_{u,k})^2 + \sum_{k=1}^{n} \rho_k^2 \right) \]  
(A.12)

where \( \rho_k \) and \( \rho_{u,k} \) represent the output and residual autocorrelations, respectively.

As an example, consider a process where the delay is 1. The variance of the estimate of \( \eta(b) \) is approximately

\[ \text{var}[\hat{\eta}(b)] \approx (4/n) \left[1 - \eta(b)\right]^2 \sum_{k=1}^{n} \rho_{u,k}^2 \]  
(A.13)

This result is quoted in Hosking (1991).

Approximations for the first two moments of \( \hat{\xi}(b) \), Equation (16), \( \xi(b) \) are derived in an identical manner to those for \( \hat{\eta}(b) \). They are

\[ \text{mean}[\hat{\xi}(b)] = \xi(b) \]  
(A.14)

\[ \text{var}[\hat{\xi}(b)] \approx (4/n) \xi(b)^2 \left(\sum_{k=1}^{n-1} (\rho_k - \rho_{u,k})^2 + \sum_{k=1}^{n} \rho_k^2 \right) \]  
(A.15)

In this case we note that the variance is proportional to the square of the mean, hence it may be appropriate to plot the logarithm of \( \hat{\xi}(b) \) as a variance stabilizing transformation.

Acknowledgements

The authors gratefully acknowledge the support of Imperial Oil Limited, and the Natural Sciences and Engineering Research Council of Canada, as well as the useful suggestions of Mike Foley.

Nomenclature

\( a_i \) = white noise driving force, Equation (2)
\( a_i' \) = white noise driving force for non-constant setpoint, Equation (16)
\( b \) = number of whole periods of delay, Equation (1)
\( B \) = backward shift operator
\( d \) = degree of differencing in disturbance model, Equation (2)
\( D_i \) = process disturbance model at time \( t \), Equation (2)
\( e_i \) = error vector, Equation (21)
\( f_i \) = \( b \)-step ahead forecast error, Equation (7)
\( f \) = frequency (cycles per sample), Equation (36)
\( F(B) \) = moving average component in closed loop process model, Equation (17)
\( G(B) \) = autoregressive component in closed loop process model, Equation (17)
\( G_{i,1}(B) \) = controller transfer function, Equation (3)
\( G_{i,2}(B) \) = controller transfer function for \( y_i \), Equation (15)
\( h_{n,1}(f) \) = spectrum of process under minimum variance control, Equation (38)
\( h_{n,1}(f) \) = spectrum of \( y_i \), Equation (36)
\( h_{n,1}(e_i) \) = spectrum of \( y_i \), Equation (42)
\( h_{n,1}(f) \) = covariance spectrum, Equation (43)
\( j \) = \( \sqrt{1} \)
\( J \) = cost function, Equation (26)
\( k \) = integral controller tuning parameter, Equation (35)
\( m \) = autoregressive model order, Equation (20)
\( n \) = sample length, Equation (22)
\( N \) = normal distribution, Equation (30)
\( S^2_n \) = residual mean square error, Equation (24)
\( S_{y_i,1} \) = estimated minimum variance at time \( t \), Equation (27)
\( S_{y_i,1} \) = sample closed loop deviation variance at time \( t \), Equation (28)
\( y_i \) = manipulated variable at time \( t \) (deviation from reference value), Equation (1)
\( U \) = numerator quadratic form, Equation (A.2)
\( V \) = denominator quadratic form, Equation (A.2)
\( X \) = regression matrix, Equation (21)
\( \tilde{y} \) = vector of mean corrected output (deviation from setpoint), Equation (21)
\( \tilde{y} \) = \( b \)-step ahead forecast, Equation (7)
\( \hat{y} \) = mean corrected output at time \( t \) (deviation from setpoint), Equation (21)
\( y_i \) = measured output time \( t \) (deviation from setpoint), Equation (8)
\( \bar{y} \) = sample mean process output (deviation from setpoint), Equation (25)
\( y_{sp} \) = process setpoint (constant), Equation (3)
\( y_{sp,1} \) = process setpoint at time \( t \), Equation (15)
\( Y_i \) = measured process output, Equation (1)
Greek symbols

\( \alpha \) = parameter vector, Equation (21)
\[ \{a_k\} \] = autoregressive parameters, Equation (19)
\( \gamma_k \) = autocovariance at lag \( k \), Equation (A.9)
\( \gamma_k \) = residual autocovariance at lag \( k \), Equation (A.9)
\( \delta(B) \) = denominator polynomial in process dynamics, Equation (1)
\( \Delta(f) \) = difference between present control and minimum variance control, Equation (39)
\( \eta(b) \) = normalized performance index, Equation (14)
\( \hat{\eta}(b) \) = estimate of normalized performance index, Equation (25)
\( \eta(b,f) \) = spectral interpretation of the normalized performance index, Equation (45)
\( \theta(B) \) = moving average component in disturbance model, Equation (2)
\( \lambda \) = recursive least squares forgetting factor, Equation (27)
\( \Lambda \) = recursive least squares weighting matrix, Equation (26)
\( \mu \) = mean of \( Y \) (no control), Equation (1)
\( \rho \) = mean process output (deviation from setpoint), Equation (12)
\( \rho \) = mean of \( Y \) (feedback control), Equation (4)
\( \xi(B) \) = performance index, Equation (13)
\( \rho_k \) = true autocorrelation at lag \( k \), Equation (30)
\( \rho_k \) = residual autocorrelation at lag \( k \), Equation (32)
\( \sigma^2_{x^2} \) = b-step ahead forecast variance, Equation (11)
\( \sigma^2_{x^2} \) = white noise variance, Equation (36)
\( \sigma^2_{\mu} \) = variance of theoretical minimum variance controller, Equation (10)
\( \sigma^2_{\mu} \) = closed loop deviation variance, Equation (11)
\( \sigma^2_{\nu} \) = variance-covariance matrix, Equation (A.9)
\( \sigma^2_{\nu} \) = variance-covariance matrix, Equation (A.9)
\( \phi(B) \) = autoregressive component in disturbance model, Equation (2)
\( \psi(B) \) = closed loop transfer function, Equation (4)
\( \psi(B) \) = closed loop transfer function for non-constant setpoint, Equation (16)
\( \psi(B) \) = b-step ahead forecast error polynomial, Equation (5)
\( \psi(B) \) = b-step ahead forecast polynomial, Equation (5)
\( \omega(B) \) = numerator polynomial in process dynamics, Equation (1)
\( \nu \) = difference operator, Equation (2)

References


Manuscript received November 22, 1991; revised manuscript received April 20, 1992; accepted for publication April 27, 1992.

THE CANADIAN JOURNAL OF CHEMICAL ENGINEERING, VOLUME 70, DECEMBER, 1992 1197