Analysis of Multivariable Controllers Using Degree of Freedom Data

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SUMMARY

Most approaches for monitoring, diagnosis and performance analysis of multivariable control loops employ time series methods and use non-parametric statistics to analyze the process inputs and outputs. In this paper, we explore the use of a discrete variable that summarizes the status of the constraint set of the controller to analyze the long run behavior of control systems. We introduce a number of waiting and holding time statistics that describe the status of this data, which we call the degree of freedom data. We demonstrate how Markov Chains might be used to model the status of the degree of freedom data. This model-based approach has the potential to provide considerable insight into the behavior of a model based control scheme with relative ease. We demonstrate the methodologies on simulated and industrial data.

KEY WORDS: multivariable control, performance assessment, Markov Chains, discrete data

1. INTRODUCTION

Monitoring, diagnosis and performance assessment of control loops is a topic of considerable academic and industrial interest. This focus is motivated by the importance that control systems have in enabling companies to achieve goals related to quality, safety and asset utilization. Reviews and critical analysis of several approaches for assessing control loop performance can be found in [1, 2, 3, 4]. Control system capability statistics based on the performance benchmark of minimum variance control for single-input single-output (SISO) systems were the initial underlying concept for much of this work. Analysis of systems involving regulation of stochastic and deterministic disturbances, setpoint tracking, extensions
to multiple-input single-output (MISO) systems (i.e., single output systems with feedforward variables) can be accommodated in this framework, and these aspects have been described in the aforementioned references. A commercial product/service for control loop assessment was described in [5].

Extensions of univariate performance assessment techniques to multiple-input single-output and to multiple-input multiple-output (MIMO) systems for general time delay systems were initially considered in [6, 7]. The primary objective in these papers was to determine the control invariant, i.e. that part of the closed loop system that is not affected by feedback. Analysis of this invariant then provides a lower bound on the performance of the control system in the mean-square error or variance sense. Recent work on analyzing multivariable control systems has focused on using impulse models and variance decompositions [11, 8]. A model predictive control framework was used to assess the performance of multivariable controllers [9, 10]. A number of the challenges in diagnosing multivariable control systems have been also investigated in [11, 12].

Methods used to analyze univariate and multivariate control systems employ non-parametric statistics and process models. Histograms, scatter plots, time-series plots, autocorrelation functions and spectral analysis are examples of the non-parametric statistics commonly encountered. Most of the process models assume that the process can be modelled by a (locally) linear transfer function plus additive stochastic or deterministic disturbance. In spite of the sophistication of the theory and software for modelling multivariate linear processes, there are many challenges in evaluating and monitoring a multivariable control system. These challenges arise primarily from: i) the requirement for a priori knowledge of the time delay structure of the process, ii) the time-varying nature of control loops which arises from the constraint handling requirements of multivariate controllers, iii) the requirement to use sophisticated identification methods to obtain meaningful estimates of the closed loop impulse weights, and iv) ensuring that the control objectives are properly incorporated in the monitoring and diagnosis strategy. There are a significant number of challenges, both theoretical and practical, in the assessment of multivariate schemes.

In this paper, we will take a different approach for the analysis of multivariable control systems that incorporate controlled variable and manipulated variable constraints. Instead of analyzing the controlled and manipulated variables, we shall examine the status of the constraint set for the control system. In this paper we will analyze a variable, that we call the degrees of freedom (DOF), which is simply the difference between the number of unconstrained manipulated variables and the number of controlled variables. A control system that has a DOF status less than zero, is one that is unable to ensure adequate regulation in all of the controlled variables, and cannot follow changing setpoints. In contrast, a control system that has a positive DOF status has flexibility for regulation and setpoint tracking to meet additional control system objectives.

Analysis of the status of the constraint set does not eliminate the need for other methods of analysis. Rather it provides us with complementary information about the performance of the control system that should be of use in explaining variance or mean square error type performance measures. A reduction in the number of degrees of freedom may not lead to inflated variance in a particular controlled variable. However, changes in the DOF status do imply that the control system is being ‘reconfigured’ in response to changes in setpoints or constraints such as valve saturation that may result from disturbances. The DOF status may also change if the tuning parameters of the control system are adjusted. Changes in the DOF
will most likely result then in changes in variances of the inputs and outputs. Consequently, one would expect that these traditional performance measures might change as the DOF change. However, the study of the DOF state of a control system is justified without reference to these traditional performance measures. The DOF status enables analysis of the changing nature of the constraints imposed on the control system. Rapidly changing constraint conditions might point the need for an analysis of the tuning parameters of the controller or of the optimizer that is determining setpoints for the controller.

In analyzing DOF data one might be interested determining the average DOF status over some time frame, i.e. how often a particular DOF status is obtained. One might also be interested in analyzing dynamic characteristics of the DOF data, such as: how long does one stay in a particular DOF state, or how long does it take for a control system to return to a particular DOF state? To enable a comprehensive analysis of the DOF status, we shall employ non-parametric statistics and model based methods. In particular we investigate the use of Markov Chains to model the DOF status. The advantage of using this type of model is that considerable insight can be achieved about the behavior of the control system with relative ease. The methods proposed in this paper are applicable to any variable that takes on a finite number of values.

The outline of the paper is as follows. First, we introduce some new variables that are derived from time spent in a particular state or the transition time between states. We analyze a sequence of simulated DOF data using summary statistics for these variables. This is followed by a brief review of Markov Chains. There are many references to this topic and we have only included a short synopsis of the topics used in this paper. The simulation example is then re-analyzed using Markov Chains. DOF data from a three output, six input model predictive controller is then analyzed using the methods proposed in this paper. Conclusions and recommendations follow.

2. NON-PARAMETRIC STATISTICS

In this section we will define several variables that relate to time spent in a particular state or the transition time between states. These are random variables and we shall use our data record to obtain samples of these derived quantities. When we are able to model the changes in states by a Markov Chain, the mean and variance of several of these variables are readily calculated from the transition probability matrix.

We assume that we have \( n \) observations of the discrete variable \( X_t, t = 1, \ldots, n \). We restrict ourselves to the situation where there are a finite number of outcomes for \( X_t \). We shall formally note the set of \( q \) possible outcomes for \( X_t \) which is known as the state space. The \( i^{th} \) element of \( S \) is denoted by \( S_i \). It is convenient to consider two types of changes of state: real changes and virtual changes \[13\]. The former involve a change in the index of the state, i.e. from state 1 to 2, whereas the latter do not involve a change in the state index, i.e. from state 1 to 1.

**Occupation times:**

- We define the number of times that the sequence is observed to be in a particular state
The occupation time is a q vector of counts whose $i^{th}$ element is defined as:

$$C_i = \sum_{t=1}^{n} 1[X_t = S_i]$$

(1)

where $1[X_t = S_i]$ is the indicator function and is defined as:

$$1[X_t = S_i] = \begin{cases} 
0 & X_t \neq S_i \\
1 & X_t = S_i 
\end{cases}$$

(2)

The elements of $C$ sum to $n$. From the occupation times, one can calculate the fractional occupation time $F_i$ in a particular state $i$, $F_i = C_i/n$.

**Waiting time statistics:**

- We define $U$ to be the sequence of elapsed times between changes in state (Only real changes are considered) and $\bar{U}$ to be an average time interval in which the sequence spends in any particular state before transiting to another state.
- We define $T_i$ to be the sequence of elapsed times in state $i$ before a transition to any other state occurs and $\bar{T}_i$ to be the average time interval in which the sequence spends in state $i$ before transiting to another state. $\bar{T}_i$ is known as the state-holding time. It is clear that $U = T_1 \cup T_2 \cup \ldots \cup T_q$.

**Transition and return time statistics:**

- We define $M_{ij}$ to be the sequence of elapsed times between entry into state $i$ and first arrival at state $j$. When computing $M_{ij}$, we include in the sequence virtual changes, i.e. we include an elapsed time value of one, whenever adjacent values of the state are at the same value. We also include sequences where a return to state $i$ occurs before state $j$ is reached. For example, if $X = \{1, 1, 2, 1, 3, 1\}$ then $M_{11} = \{1, 2, 2\}$ and $M_{13} = \{4\}$. $M_{ij}$ is the sequence of return time values. The average of these times, $\bar{M}_{ij}$, represents the mean return time from state $i$ to state $j$ and is sometimes referred to as the mean passage time.
- We define $R_{ij}$ to be the sequence of shortest elapsed times between an exit from state $i$ and first arrival at state $j$. Only real changes are considered and we do not include in our sequence those times where a return to state $i$ occur before state $j$ is reached. If $X = \{1, 1, 2, 1, 3, 1\}$ then $R_{11} = \{2, 2\}$ and $R_{13} = \{1\}$. We shall call $R_{ij}$ the sequence of transit time values. The average of these values, $\bar{R}_{ij}$, represents the mean transit time from state $i$ to state $j$.

It is straightforward to display the discrete probability density functions of any of these quantities in a matrix plot, or to compute summary statistics such as means and medians, or perform a spectral analysis to look for periodicities. If desired, one can fit a number of empirical waiting time models to this data.

To illustrate, suppose that:

$$X = \{1, 1, 3, 1, 2, 2, 3, 3, 1, 1, 2, 1, 3, 1, 1, 1, 2, 2, 3, 1\}$$

(3)

Then:

$$C = \{10, 5, 5\}$$

(4)
The sequences of waiting times are:

\[ U = \{2, 1, 1, 2, 2, 1, 1, 3, 2, 1, 1\} \quad (5) \]

\[ T_1 = \{2, 1, 2, 1, 3, 1\} \quad T_2 = \{2, 1, 2\} \quad T_3 = \{1, 2, 1, 1\} \quad (6) \]

The sequences of return time values are:

\[ M_{11} = \{1, 2, 5, 1, 2, 2, 1, 4\} \quad M_{12} = \{4, 2, 5\} \quad M_{13} = \{2, 3, 4, 5\} \]
\[ M_{21} = \{4, 1, 3\} \quad M_{22} = \{1, 5, 6, 1\} \quad M_{23} = \{2, 2, 2\} \quad (7) \]
\[ M_{31} = \{1, 2, 1, 1\} \quad M_{32} = \{2, 4, 4\} \quad M_{33} = \{4, 1, 5, 6\} \]

The sequences of transit time values are:

\[ R_{11} = \{2, 5, 2, 2, 4\} \quad R_{12} = \{1, 1, 1\} \quad R_{13} = \{1, 3, 1, 3\} \]
\[ R_{21} = \{3, 1, 2\} \quad R_{22} = \{5, 6\} \quad R_{23} = \{1, 2, 1\} \quad (8) \]
\[ R_{31} = \{1, 1, 1, 1\} \quad R_{32} = \{2, 3, 4\} \quad R_{33} = \{4, 5, 6\} \]

Simulation example:

In this section we demonstrate the use of the derived quantities on a simulation example of a 4 state system, corresponding to degrees of freedom equaling \{-1, 0, 1, 2\}. Details on how we generated the system are discussed in the next section. We will assume that 2000 observations are available for analysis.

Figure 1 shows a time series of the first one hundred observations. States 2 and 3 are occupied more frequently than states 1 and 4. We calculate the \( \bar{U} = 1.5 \). This implies that the system spends 1.5 time interval in any particular state before transiting to another state. In Table I we summarize the occupation times and fractional occupation times. In Table II we show the mean of the state holding time sequences \( T_i \). Tables III and IV show the mean and mode of the return times and transit times. We shall defer comment on the mean return times until the next section. For the mean transit time, we note that when the system is in State 1 (DOF=-1), 1.28 time intervals elapse, on average, before the system moves to State 2 (DOF=0) once it leaves State 1. As well, once the system leaves State 1, on average 9.54 time intervals elapse before it returns to this state.

We notice that the mean return times, \( \bar{M}_{ii} \), are close to the inverse of the fractional occupation time. This is not a coincidence, and this will be discussed further in the next section.

3. Markov Chain Analysis

In the previous section we defined a number of terms to describe the behavior of the degree of freedom data. We deliberately chose a non-parametric approach as this does not require construction of a model to describe the data. This method of analysis, while easy to use and clearly useful, does not allow one to analyze new scenarios by extrapolating beyond the current data set. In this section we will use a number of model-based approaches to analyze \( X_t \). This will enable analysis of DOF data beyond what can be achieved through descriptive statistics alone.
Time series models are widely used to aid in process monitoring and diagnosis. The theory is well developed and software readily available. A number of methods have been developed to enable modelling and prediction of integer valued time series, such as Integer Autoregressive Models (INAR), or Poisson Autoregressive Models (PAR). An overview of these and related approaches for modelling integer valued time series data is provided in [14]. The use of time series models is most appropriate when one is interested in making predictions for future values of the time series. In the case of the degree of freedom data, this is not our primary interest as we are interested in the long-term behavior of the system. The use of Markov Chains would appear to be more appropriate. There is a vast literature on the theory and application of Markov Chains. We will only include the necessary material for our analysis. Some accessible references on Markov Chains include in [15, 16, 17]. More rigorous treatments can be found in [13, 18, 19]. Many of the properties of Markov Chains can be explained from an eigenvalue / eigenvector perspective and the seminal reference for this approach is [20]. Finally, the parameters of Markov Chains must often be estimated from data. Methods for estimating the parameters, assessing their uncertainty and hypothesis tests concerning the order of the Markov Chain are given in a series of classic papers described in [21, 22, 23].

3.1. Basic concepts and definitions

We will confine ourselves to discrete homogenous Markov Chains – that we are assuming that the parameters of the system are time invariant. We recall that we have restricted our attention to the case where we have q possible outcomes at each point in time. Let us suppose that we are able to describe the probability of being in any one of these q states at time t=0 by the vector \( \pi_0 \). That is, \( (\pi_0)_j = \text{Probability}(X_0 = S_j) \). \( \pi_0 \) is known as the initial distribution. The elements of \( \pi_0 \) sum to one. We are now interested in calculating the probability of being in state \( j \), at \( t=1,2,\ldots \). In Markov Chain analysis, the probabilistic evolution of the system is defined through the \( q \times q \) transition probability matrix \( P \). The \( ij^{th} \) element of \( P, p_{ij} \), is the probability of transition from state \( i \) to state \( j \) in one step. Clearly \( p_{ij} \geq 0 \) for all \( i \) and \( j \) and each row of \( P \) sums to one. \( P \) is known as a stochastic matrix. Given \( \pi_0 \) and \( P \), the distribution of probabilities among the \( q \) states at \( t=1 \), is:

\[
\pi^T_1 = \pi^T_0 P
\]  

(9)

Given the structure of \( \pi_0 \) and \( P \), the elements of \( \pi_1 \) sum to one. The Probability\( (X_1 = S_j) \) is the \( j^{th} \) element of \( \pi_1 \). In Markov Chain analysis, the distribution of probabilities among the \( q \) states at \( t = k \) is given by:

\[
\pi^T_k = \pi^T_{k-1} P
\]

\[
= \pi^T_0 P^k
\]

(10)

A key feature of the evolution of the probabilities relates to the following identities that underline the development of this equation:

\[
\text{Probability}\{X_1 = S_j \mid X_{t-1} = x_{t-1}, X_{t-2} = x_{t-2}, \ldots X_0 = x_0\}
\]

\[= \text{Probability}\{X_1 = S_j \mid X_{t-1} = x_{t-1}\}\]

(11)

where \( \{x_{t-j}, j = 1, \ldots, k\} \in S \). This is known as the memoryless property of Markov processes. This property states that all of the system information prior to time \( t \) is reflected in the distribution of probabilities at time \( t-1 \). Equation (11) defines the probability rules for a 1\textsuperscript{st} order Markov process. Additional information about the past history of the process is of no
value in determining the evolution of the future distribution of probabilities. The incorporation
of additional history can be accommodated by consider Markov Chains of higher order. We
shall demonstrate this approach when we examine our industrial data.

In spite of its simple structure, systems that evolve according to the aforementioned Markov
behavior admit a time evolution that has many interesting features. We shall summarize a few
key items.

3.2. Ergodic Markov chains

We say that $i \rightarrow j$ if it is possible in a finite number of steps, to get from state $i$ to $j$. If it
is also possible to get from state $j$ to $i$ in a finite number of steps, $j \rightarrow i$, then we say that $i$
communicates with $j$. This is denoted by $i \leftrightarrow j$. For distinct states, $i$ and $j$, a necessary and
sufficient condition for $i \rightarrow j$ is that $p_{ij}^m = (P^m)_{ij} > 0$ for some $m \geq 0$ [16]. If $i \rightarrow j$ and
$j \rightarrow k$ then $i \rightarrow k$. A closed class is one for which there is no escape. If the state transition
matrix contains an element $p_{kk} = 1$, then state $k$ is closed. A system in which there are no
closed classes is called irreducible. A necessary and sufficient condition for a Markov Chain
to be irreducible is that there exists an integer $m$ such that $p_{ij}^m > 0$ for all pairs $i$ and $j$.
A constructive method for ascertaining whether a Markov Chain is irreducible using matrix
multiplication was provided in [17]. An eigenvalue analysis of the state transition matrix can
also be used to check for irreducibility.

For an irreducible Markov Chain, one can imagine a situation where one returns to a specific
state $k$ at regular time intervals which is a multiple of an integer $d$, $d \geq 2$, even though the
system is inherently a stochastic system. A system in which there is periodic return to a
specific state is said to be aperiodic [16]. A homogenous Markov Chain that is irreducible and
aperiodic is ergodic. A necessary and sufficient condition for a Markov Chain to be ergodic
(assuming that it is homogenous) is that there only be one eigenvalue of $P$ whose magnitude
is unity [20]. This provides a constructive test for ergodicity. An eigenvalue analysis can also
be used to check for periodicity.

3.3. Equilibrium distribution

Consider the case where a Markov Chain process starts with some initial distribution and is
allowed to run for a ‘long time’. We know that:

$$\pi_k^T = \pi_0^T P k = \pi_{k-1}^T P$$

Let us suppose that as $k \rightarrow \infty$, $\pi_k = \pi_{k-1} = \pi$. In other words,

$$\text{Probability}(X_k = S_j) \rightarrow (\pi)_j \quad k \rightarrow \infty \quad \text{for all } j$$

The probability of being in a specific state is not affected by the initial state. The long
run distribution that satisfies this property is known as the equilibrium distribution and is
usually denoted by $\pi$. A sufficient condition that ensures the existence of a unique equilibrium
distribution, is that $P$ be ergodic [16]. Furthermore :

$$p_{ij}^m = (P^m)_{ij} \rightarrow (\pi)^T \quad m \rightarrow \infty \quad \text{for all } j$$

One might suspect that for an ergodic Markov Chain, that a relationship would exist between
the occupation times and the equilibrium distribution. Indeed this is the case. Asymptotically
\[
\frac{C}{n} \sim MVN(\pi, \frac{V}{\sqrt{n}}) \quad (15)
\]

where MVN is an acronym for Multivariate Normal. Expressions for the variance-covariance matrix, \(V\), are very complicated functions of \(P\) and are not particularly useful.

### 3.4. Mean return times

Starting in state \(i\), one is often interested in the mean return time to get to state \(j\). Denote this mean return time by \(\bar{M}_{ij}\). It is known that the solutions for \(\bar{M}_{ii}\) is given by:

\[
\bar{M}_{ii} = \frac{1}{\pi_i} \quad (16)
\]

The matrix of mean return times is \[18\]:

\[
\bar{M} = (I + Z + EZ_{\text{diag}})D \quad (17)
\]

where \(E\) is a \(q \times q\) matrix composed of 1’s, \(D\) is a diagonal matrix whose elements are \(1/\pi_i\), and \(Z\), known as the fundamental matrix, is given by:

\[
Z = (I - (P - \Pi))^{-1} \quad (18)
\]

\(\Pi\) is a matrix whose rows are equal to the equilibrium distribution \(\pi^T\). \(Z_{\text{diag}}\) denotes the diagonal elements of the fundamental matrix. Variance expressions for mean return times can also be developed \[13, 18\].

### 3.5. State holding time

We defined \(T_i\) to be the sequence of times that elapse in state \(i\) before a transition to any other state \(j, j \neq i\) occurs. For a process described by a Markov Chain, this sequence has a geometric distribution with parameter \(1 - p_{ii}\). The mean of \(T_i\) is \(1/(1 - p_{ii})\) and the variance of \(T_i\) is \(p_{ii}/(1 - p_{ii})^2\).

It is also possible to develop an expression for the expected (mean) time spent in state \(i\) before a transition to state \(j\). When \(P\) is ergodic this is given by \(\pi_j/\pi_i\) \[18\]. Reference \[13\] provides a number of interpretations for the state holding time, mean return time and related concepts from a system perspective.

### 3.6. Estimating the transition probability matrix

Define \(C_{ij}\) to be the number of occurrences in the data record where the process in state \(j\) at time \(t\), and state \(i\) at time \(t-1\). One could estimate the elements of \(P\) by:

\[
\hat{p}_{ij} = \frac{C_{ij}}{\sum_{j=1}^{n} C_{ij}} = \frac{\sum_{t=2}^{n} 1 \left[X_t = S_j\right] 1 \left[X_{t-1} = S_i\right]}{\sum_{t=2}^{n} 1 \left[X_{t-1} = S_i\right]} \quad (19)
\]

\(\hat{p}_{ij}\) is the maximum likelihood estimate for \(p_{ij}\) \[21, 23\]. \((\hat{p}_{ij} - p_{ij})\) is asymptotically normally distributed with mean zero and variance \(p_{ij}(1-p_{ij})/\sqrt{C_i}\). Expressions have also been developed for the covariance between \(p_{ij}\) and \(p_{kl}\) \[23\]. In computing variance estimates, one doesn’t know...
the true values for the probabilities and it is necessary to replace the true values by the maximum likelihood estimates.

An alternate approach for estimate the transition probability values is to form a window estimate. In this method, we propose to estimate to divide the data into \( D \) blocks, each of length \( n/D \). The elements of \( P \) are estimated using data record \( d = 1, \ldots, n/D \). From each of the individual estimates \( \hat{P}(d) \) and overall average for \( \hat{P} \) is calculated. A byproduct of this approach is that obtains an empirical estimate of the variance for the elements of \( \hat{P} \). Some judgement must be used when to choosing the window length. This approach also enables one to test for homogeneity in the data.

3.7. Testing homogeneity and model order

Thus far we have assumed that the process is 1\(^{st}\) order Markov and is homogenous (i.e. system parameters are time invariant). A Likelihood Ratio (LR) test for homogeneity is outlined in [24]. Divide the data record into \( D \) segments (these need not all be of the same length). Calculate the overall number of transitions from state \( i \) to state \( j \), \( C_{ij} \) and the number of transitions from state \( i \) to state \( j \) in each data segment, \( C_{ij}(d), \) \( d = 1, \ldots, D \). To test the null hypothesis that the process is homogenous (assuming that it is first order) we compute:

\[
LR = 2\sum_{d=1}^{D} \sum_{i,j} C_{ij}(d) \ln \left( \frac{\hat{p}_{ij}(d)}{\hat{p}_{ij}} \right)
\]  

(20)

This value is compared to the \((1 - \alpha)\) critical point of the chi-squared distribution with \( D(q-1)^2 \) degrees of freedom. If \( LR \) exceeds this value, then one rejects the null hypothesis at the prescribed confidence level.

This approach can also be used to construct a test to locate change points in a sequence of data. The sequence is initially divided into two sections of length \( s \) and \( n = 2s \). If the null hypothesis is accepted, \( s \) is incremented and the likelihood ratio computed for these two blocks. The procedure is continued until a change point is detected or one concludes that the entire set is homogenous. Since multiple comparisons are being made, care must be taken in selecting the confidence level to ensure an overall confidence level for the entire test. Details on this and guidelines for selecting \( s \) are outlined in [24].

We may wish to test the hypothesis that the process is second order. A likelihood ratio test for this has been proposed [21, 23]. Define \( C_{ijk} \) to be the number of times that the process is in state \( k \) at time \( t \), state \( j \) at time \( t-1 \) and state \( i \) at time \( t-2 \):

\[
C_{ijk} = \sum_{t=3}^{n} \mathbf{1}[X_t = S_k] \mathbf{1}[X_{t-1} = S_j] \mathbf{1}[X_{t-2} = S_i]
\] 

(21)

Define the second order Markov transition probability as \( p_{ijk} \). Then the quantity

\[
LR = 2\sum_{i,j,k} C_{ijk} \left( \frac{\hat{p}_{ijk}}{\hat{p}_{jk}} \right)
\] 

(22)

is distributed as a chi-squared variable with \( q(q-1)^2 \) degrees of freedom where:

\[
\hat{p}_{ijk} = \frac{C_{ijk}}{\sum_{i,j} C_{ijk}}
\] 

(23)
and \( \hat{p}_{jk} \) is the \( jk \)th element of the estimated transition probability matrix calculated previously. To test the null hypothesis that the process is generated by a first order Markov process against the second order alternative, one computes \( LR \), and compares this to the \((1 - \alpha)\) critical point of the chi-squared distribution with \( q(q - 1)^2 \) degrees of freedom. If \( LR \) exceeds this value, then one rejects the null hypothesis at the prescribed confidence level. One is assuming of course that the data is homogenous.

A number of other statistical tests for individual parameters of the transition probability matrix and blocks of these parameters have been also developed in [21, 23] along with asymptotic results for the distribution of the equilibrium distribution.

3.8. Simulation example

Consider the Markov Chain defined by the transition probability matrix:

\[
P = \begin{pmatrix}
0.2 & 0.7 & 0.1 & 0 \\
0.2 & 0.2 & 0.5 & 0.1 \\
0.1 & 0.2 & 0.5 & 0.2 \\
0 & 0.1 & 0.7 & 0.2
\end{pmatrix}
\]  

This transition probability matrix was used to generate the data in the previous section. The eigenvalues of \( P \) are \( \{1, 0.3646, -0.1, -0.1646\} \). We readily establish that \( P \) is irreducible and aperiodic. The equilibrium distribution is given by:

\[
\pi^T = \begin{bmatrix}
0.121 & 0.246 & 0.482 & 0.151
\end{bmatrix}
\]  

The mean state holding times are given by:

\[
\bar{T}^T = \begin{bmatrix}
1.25 & 1.25 & 2 & 1.25
\end{bmatrix}
\]  

Finally, the matrix of mean returning times:

\[
\bar{M} = \begin{pmatrix}
8.22 & 1.80 & 3.21 & 8.94 \\
8.88 & 4.07 & 2.24 & 7.80 \\
10.00 & 4.40 & 2.07 & 6.91 \\
11.11 & 5.10 & 1.53 & 6.61
\end{pmatrix}
\]

These values can be compared to the sampled values shown in Tables II - IV. We note that we do not have a closed form expression for the mean transit times.

To estimate \( P \) we use blocks of 2000, 1000 and 500 observations. To summarize the data, we shall report the estimates of \( P \), the equilibrium distribution, mean holding times, and average of the standard deviation of the individual estimates. This is computed as \( \sqrt{\frac{1}{q} \sum_{i,j} \hat{V}_{ij}} \), where \( V_{ij} \) is the \( ij \)th variance estimate for \( \hat{p}_{ij} \). From Table V we see that all three methods give similar results for \( P \) and the equilibrium distribution. We note that the theoretical asymptotic variance estimates are considerably larger than those determined from the block estimates. When the likelihood ratio test is applied to this set of data, we accept the null hypothesis that the data is homogenous using 4 blocks and 2 blocks. We also accept the null hypothesis that a Markov Chain of order 1 generates the process.

Having estimated \( P \), we can directly calculate a number of return time and waiting time statistics using the equations provided above. One could also calculate these quantities directly
leading to the question why use Markov Chains rather than the observed quantities directly? By using a Markov Chain, one can simulate the process and explore the effect that possible changes might have on equilibrium distribution etc. In addition, one is able to investigate the evolution of the states from an initial distribution through Equation (12). For example, let us speculate on the effects of some process changes that might change the last row of the estimated transition probability matrix (1 block of 2000) from:

\[
\begin{bmatrix}
0 & 0.092 & 0.715 & 0.193 \\
0 & 0.092 & 0.408 & 0.500
\end{bmatrix}
\]  

(28)

We are supposing that it is possible to modify the controller tuning parameters or some other aspect of the process so that the probability of staying in State 4 (DOF = 2) changes from 0.193 to 0.500. This seems like a rather large change, and we are interested in determining the potential effect of this on the equilibrium distribution. When we do this, we find that the modified equilibrium distribution changes from:

\[
\hat{\pi}^T = \begin{bmatrix}
0.127 & 0.244 & 0.475 & 0.154
\end{bmatrix}
\]

\[
\hat{\pi}^T_{\text{mod}} = \begin{bmatrix}
0.117 & 0.232 & 0.426 & 0.225
\end{bmatrix}
\]  

(29)

We have used a circumflex to denote that the equilibrium distribution is computed from the estimated transition probability matrix. There has been a small shift in the fraction of time spent in States 3 and 4, with an obvious shift into State 4 and only modest changes spent in States 1 and 2.

4. INDUSTRIAL EXAMPLE

We have 20,000 observations from an industrial process of the DOF status of a multivariable control system collected on a one-minute basis. We have three controlled variables and six manipulated variables. During the period of data collection the state space \( S = \{ -2,-1,0,1,2,3 \} \).

There is no theoretical reason why the DOF data should be described by a Markov process. As with many empirical models, we know that they are useful, even when they are theoretically difficult to justify.

4.1. Basic description and analysis

To enable a comparison of the behavior of the control system, we divide our data into two blocks of 10,000 and analyze each one separately. We call these blocks A and B. The occupation times and fraction of time spent in each state for blocks A and B are shown in Table VI. Table VII provides a summary of the transition times, including the values in the time sequence for the state having the longest transition times, mean holding time, mean return times and mean transit time \( \bar{R}_i, i = 1, \ldots, p \). In each of these tables there are two lines for each variable. The first describes the raw data. The second line describes the statistics of a smoothed or filtered sequence. The motivation for smoothing the original data is the observation that there appears to be ‘chatter’ in the data. That is one will encounter a string of fixed states, following by a
transition to a new state that lasts one sampling interval, followed by a string of the previous
states. To filter the data, the single state transitions just described were replaced with the
state value on either side of the one time interval state change. All transitions to DOF = -2
were retained as these are rare events and warrant investigation.

A comparative analysis of the smoothed and raw data within each block shows only small
changes in the occupation time. Not surprisingly, the mean and mode of the transition times
have increased for the smoothed data. Removal of the chatter has lengthened the mean return
times and transit times. The subsequent analysis is all based on the smoothed data.

There are a number of differences between these blocks of data. First, in Block B, the system
spends more time in State 6 (DOF = 3) and less time in State 4 (DOF = 1) compared to Block
A. In both cases, the time spend in States 4–6 (DOF = 1–3) is nearly identical. We see another
reflection of this in the mean state holding times. We also notice that the mean return times
$M_{ii}$ are close to the inverse of the fractional occupation time. There is a discrepancy in mean
return times for State 1 (DOF = -2), however there are very few occurrences of this state and
the uncertainty in the summary statistics is very high.

We have tabulated the mean transit time $\bar{R}_{ij}$ for both blocks. The interpretation of this
summary statistic for Block A is as follows. If constraint set had DOF = 1, and the constraint
set was changing, we would find on average that the system would return to this constraint
condition in 29.5 minutes, would transition to DOF = 2 in 2.9 minutes and to DOF = 3 in
8.5 minutes. The transition to a condition where less control is exerted, i.e., DOF $\leq$ 0, takes
more time than a transition to a position where more control is exerted for a state change of
the same magnitude, i.e. the transit time for increase in DOF of 2 units is less than the transit
time for a 2 unit decrease in DOF.

An interesting feature is that for both blocks, the mean time between transitions is 8.9 and
11.9 minutes. However, there are some very long runs in State 6 where the constraint set is
not changing. This significant different can also result from different disturbances or setpoints
that are required during the two time periods. The reader is directed to [25] where a more
comprehensive analysis over many time blocks was undertaken.

It would be rather interesting to correlate other measures of performance with the changes
in state. A traditional analysis of performance using variance should be undertaken with care
when the DOF status is changing rapidly. The changing constraints may result in limited
segments of data with no changes in DOF. If the changing DOF status is ignored, the
assumptions required to undertake an analysis that is based on the principle of superposition
of linear dynamics and disturbances are invalidated.

4.2. Estimating the transition probability matrix

We used the methods described previous to estimate $P$. There was very little difference between
blocking using 5000 observations and a single block of 10,000 observations, and so we used
the latter. For both blocks of data, we accept the null hypothesis that the data is homogenous
using $D = 2$, i.e. blocks of 5000. When we subdivided the data into smaller blocks, we rejected
the null hypothesis that the data is homogenous. That means we cannot use one invariant $P$
matrix to present the whole process behavior. The estimated transition probability matrices
and the associated standard deviations of the estimates (using the results of [23] described
in Section [3.6]) are reported in Table [VIII] for both blocks of data. The transition probability
matrix for both blocks of data is irreducible and aperiodic. Also shown in this table is the
equilibrium distribution and state holding times.

Several comments are in order. For both blocks we notice that \( \hat{p}_{i(i+1)} \geq \hat{p}_{i(i-1)}, i = 2, \ldots, 5 \). This means that when the system is in states 2-5, it is more likely to transit to a state where there are more degrees of freedom available for control. The other feature that we notice is that when the degrees of freedom change, it is most likely that only a single degree of freedom change will occur. We note that since \( \hat{R}_{41} \) is non-zero and \( \hat{p}_{41} \) is zero, it is not possible to transit from State 4 to State 1 in one step. The observations that we made on the empirical mean state holding times are reflected in the elements \( \hat{p}_{44}, \hat{p}_{55}, \hat{p}_{66} \). These values are close to the empirical values calculated from the occupation times.

The chi-squared test for a Markov model of order 1, fails at the 95% confidence level. The consequences of this are: i) the non-parametric statistics, and their interpretation are still valid, ii) the calculation of the transition probability matrix and its interpretation is still valid, iii) the estimates of the uncertainty in the elements of the transition probability matrix are not true when using the formulas given in [23], and iv) derived quantities from the transition probability matrix are not valid. Although we reject the null hypothesis that the data are generated from a Markov process of order one, we have still gained considerable insight into the behavior of the process by analyzing the transition probability matrix.

A more comprehensive analysis of this data is described in [25]. It is shown that the ‘failure’ to pass the 1st or Markov property arises from a few small subsets of data. If the data were divided into blocks of 1000 observations, then 3 blocks in 20 fail the null hypothesis at the 95% confidence level and only one block fails the chi-squared test at the 99% confidence level.

4.3. Contracting and expanding the state space

It may happen that a coarser analysis of the constraint set is required. This might be desirable when there are a large number of input and output variables. In these instances one might define the states as groups of variables, say all DOF values between \(-5\) and \(-1\), 0, 1-4, 5-8, etc. We shall briefly illustrate this concept using the industrial data. We now define the following states: State 1: DOF<0, State 2: DOF=0 and State 3: DOF>0. These states have a clear physical interpretation. We will not show the non-parametric statistics. The estimated transition probability matrices and associated equilibrium distributions and mean state holding times derived from the transition probability matrices are shown in Table IX for both blocks of data. An interesting feature of the lumped analysis is that the state holding times for the lumped states are larger for block B even though the equilibrium distributions are nearly the same. We note that the state holding times will be sensitive to the values of \( \hat{p}_{ii} \) whenever this value exceeds 0.9.

We reject the null hypothesis that the data is generated by a 1st order Markov process at the 95% confidence level. We now investigate construction of a higher order Markov model. We shall illustrate this using the contracted degree of freedom data. The second order Markov model requires that we be able to evaluate \( \text{Probability}(X_{t+1} = S_k \mid X_t = S_j, X_{t-1} = S_i) \) for all states \( i, j \) and \( k \) and a model that allows for the evolution of an initial distribution. It is well known that a second order Markov process can be formulated as a first order Markov process with a state space with \( q^2 \) elements [18]. We note that:

\[
\begin{align*}
\text{Probability}(X_{t+1} = S_k \mid X_t = S_j, X_{t-1} = S_i) \\
= \text{Probability}(X_{t+1} = S_k, X_t = S_j \mid X_{t-1} = S_i)
\end{align*}
\] (30)
The last equation is of the form:

$$\text{Probability}(W_{t+1} = \tilde{S}_{j'} \mid W_t = \tilde{S}_{i'})$$

(31)

where $W_t$ is a new integer variable that has the unique value $\tilde{S}_{i'}$ whenever $X_t = S_{j} \& X_{t-1} = S_{i}$. $i'$ is an index that maps $(i, j)$ in the original variable definition into a state number, for the new variable, i.e. $i = 1, j = 1, i' = 1, i = 2, j = 1, i' = 2$. The general term is $i' = q(j-1) + i$, $i, j = 1, \ldots, q$. A suitable definition for the new variable $W_t$ is:

$$W_t = q(j-1) + i, \quad X_t = S_{j} \& X_{t-1} = S_{i}, \quad i, j = 1, \ldots, q$$

(32)

A first order Markov Chain with the variable $W_t$ corresponds to a second order Markov Chain in the variable $X_t$. The state space for $W_t$ is of dimension $q^2$. We also observe that $\tilde{P}$, the transition probability matrix for $W_t$ is sparse. For example, it is not possible to transit from a state where $(X_t = S_1 \& X_{t-1} = S_2)$ to a state where $(X_{t+1} = S_1 \& X_t = S_2)$. The only elements of $\tilde{P}$ matrix, $\tilde{p}_{ijj'}$ that are non-zero are located at:

$$\tilde{p}_{ijj'} = \tilde{p}_{(ij)(jk)} \begin{cases} 
  i, j, k = 1, \ldots, q \\
  i' = q(j-1) + i \\
  j' = q(k-1) + j 
\end{cases}$$

(33)

We can readily verify that at most $q^3$ of the $q^4$ elements of $\tilde{P}$ are non-zero.

To illustrate these methods using the contracted DOF data for industrial data. Defining $W_t$ as in Equation (32), we estimate the transition probability matrix for this variable. This value is reported in Table XI along with the equilibrium distribution and the mean holding times. Let us denote the three original states at time $t$ as Low, Neutral, High $\{L_t, N_t, H_t\}$.

We can give the following interpretation to the first four elements in the first row of the state transition matrix. $\tilde{p}_{11}$ describes the probability of transition from a $L_tL_{t-1}$ state to a $L_{t+1}L_t$ state (permissible), $\tilde{p}_{12}$ describes the probability of transition from a $L_tL_{t-1}$ state to a $L_{t+1}N_t$ state (not possible), $\tilde{p}_{13}$ describes the probability of transition from a $L_tH_{t-1}$ state to a $L_{t+1}H_t$ state (not possible) and $\tilde{p}_{14}$ describes the probability of transition from a $L_tL_{t-1}$ state to a $N_{t+1}L_t$ state (possible). From the previous discussion, we also note that $\tilde{p}_{11}$ also describes the probability of transition from a state where $(X_t = L, X_{t-1} = L)$ to $(X_{t+1} = L, X_t = L, X_{t-1} = L)$. The other state definitions follow immediately.

Whereas we rejected the null hypothesis that a Markov process of order one generated the compressed data, we now accept the null hypothesis that data is generated by a Markov process of order two at the 95% confidence level. (To construct this test, we apply the likelihood ratio test to the 9 state system described in the previous paragraph). The estimate transition probability matrix, equilibrium distribution and state holding times are shown in Table XI. We note that the transition probability matrix is sparse. Several probability values are very small. The holding times are greater than one for States 1, 5 and 9. This is not unexpected as the diagonal elements of $\tilde{P}$ for the other states are zero.

5. CONCLUSIONS

Most analysis of multivariate control systems for the purposes of monitoring, diagnosis and performance assessment, employ a combination of non-parametric and models to analyze the
process inputs and outputs. In this paper, we have examined the status of the constraint set of a multivariable control system using waiting time and transition time sequences. We are able to use a variety of non-parametric statistics to gain insight into the behavior of the control system. The advantage of the non-parametric statistics is their simplicity of calculation, ease of interpretation and lack of requirements about the nature of the underlying process. We have also examined the use of Markov Chains to understand the long-term behavior of the constraint set. Considerable insight into the long run behavior of the control system can be obtained with relative ease. The assumptions underlying the use of a Markov Chain must be tested, and a method for checking the model order was provided. The results in this paper represent an initial approach analyzing DOF status. Further research should involve undertaking a comprehensive simulation study of the interaction between the constraint set and other performance measures on a benchmark process.

One of the fundamental assumptions underlying the use of the constraint set data is that we have a representative set of data from which to base our analysis. It can be expected that the system generating the set of constraints will change over time due to changes in the controller parameters, shifts in process operating regime (such as switches in feedstock), different types of disturbances etc. These shifts do not invalidate the use of these methods. Rather, the use of this analysis provides a concise summary of the effect of all of these changes on the constraint set. The calculation methods proposed are very simple and could easily be incorporated in an analysis package that is used to routinely schedule performance analysis. Even though the calculations are straightforward, practical applications require development of suitable graphical interfaces.

Finally, we would note that the type of analysis proposed in this paper could be applied to a wide variety of analysis problems on mixtures of discrete and continuous variables. The definitions of states are quite flexible. One can readily define states that reflect only continuous variables by stratifying the continuous values (low, midrange, high), or defining states that combine attributes of two variables into a discrete value.

ACKNOWLEDGEMENTS

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REFERENCES


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**Figure 1. Sample Evolution of States: Simulation Example**

**Table I. Occupation Time for Simulated Data**

<table>
<thead>
<tr>
<th></th>
<th>State = 1</th>
<th>State = 2</th>
<th>State = 3</th>
<th>State = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Occupation Time</td>
<td>$C_i$</td>
<td>254</td>
<td>490</td>
<td>949</td>
</tr>
<tr>
<td>Fractional Occupation Time</td>
<td>$F_i$</td>
<td>0.127</td>
<td>0.245</td>
<td>0.4745</td>
</tr>
<tr>
<td>Inverse of $F_i$</td>
<td>$1/F_i$</td>
<td>7.87</td>
<td>4.08</td>
<td>2.11</td>
</tr>
</tbody>
</table>
Table II. Mean of Holding Time Sequence $\bar{T}_i$ for Simulated Data

<table>
<thead>
<tr>
<th></th>
<th>State = 1</th>
<th>State = 2</th>
<th>State = 3</th>
<th>State = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Holding Time $\bar{t}_i$</td>
<td>1.3</td>
<td>1.2</td>
<td>1.9</td>
<td>1.2</td>
</tr>
</tbody>
</table>

Table III. Mean and Mode of Return Time Sequence $\hat{M}_{ij}$ for Simulated Data

<table>
<thead>
<tr>
<th>$\hat{M}<em>{ij}$ (Mode $M</em>{ij}$)</th>
<th>State = 1</th>
<th>State = 2</th>
<th>State = 3</th>
<th>State = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOF = -1</td>
<td>7.79 (4)</td>
<td>1.60 (1)</td>
<td>3.32 (3)</td>
<td>8.34 (6)</td>
</tr>
<tr>
<td>State=2 DOF=0</td>
<td>8.38 (5)</td>
<td>4.08 (3)</td>
<td>2.29 (1)</td>
<td>7.31 (5)</td>
</tr>
<tr>
<td>State=3 DOF=1</td>
<td>9.43 (7)</td>
<td>4.42 (3)</td>
<td>2.11 (2)</td>
<td>6.68 (5)</td>
</tr>
<tr>
<td>State=4 DOF=2</td>
<td>10.78(9)</td>
<td>5.23 (4)</td>
<td>1.51 (1)</td>
<td>6.51 (4)</td>
</tr>
</tbody>
</table>

Table IV. Mean and Mode of Transit Time Sequence $\hat{R}_{ij}$ for Simulated Data

<table>
<thead>
<tr>
<th>$\hat{R}<em>{ij}$ (Mode $R</em>{ij}$)</th>
<th>State = 1</th>
<th>State = 2</th>
<th>State = 3</th>
<th>State = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOF = -1</td>
<td>9.54 (6)</td>
<td>1.28 (1)</td>
<td>2.31 (2)</td>
<td>4.27 (3)</td>
</tr>
<tr>
<td>State=2 DOF=0</td>
<td>3.03 (1)</td>
<td>4.77 (4)</td>
<td>1.21 (1)</td>
<td>2.32 (2)</td>
</tr>
<tr>
<td>State=3 DOF=1</td>
<td>1.51 (1)</td>
<td>1.65 (1)</td>
<td>3.15 (2)</td>
<td>1.46 (1)</td>
</tr>
<tr>
<td>State=4 DOF=2</td>
<td>4.28 (4)</td>
<td>3.04 (3)</td>
<td>1.23 (1)</td>
<td>7.84 (6)</td>
</tr>
</tbody>
</table>

Table V. Estimate of the Transition Probability Matrix and Equilibrium Distribution for Simulated Date

<table>
<thead>
<tr>
<th></th>
<th>4 Blocks of 500 Observations</th>
<th>2 Blocks of 1000</th>
<th>1 Blocks of 2000</th>
</tr>
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<tbody>
<tr>
<td>$P$</td>
<td>$\hat{P}$</td>
<td>$\hat{P}$</td>
<td>$\hat{P}$</td>
</tr>
<tr>
<td></td>
<td>0.200 0.700 0.100 0.000</td>
<td>0.200 0.728 0.072 0.000</td>
<td>0.205 0.720 0.075 0.000</td>
</tr>
<tr>
<td>$\pi^P$</td>
<td>0.200 0.200 0.000 0.500 0.100</td>
<td>0.210 0.181 0.514 0.095</td>
<td>0.210 0.182 0.512 0.096</td>
</tr>
<tr>
<td>$\pi^T$</td>
<td>0.121 0.246 0.482 0.151</td>
<td>0.126 0.246 0.475 0.153</td>
<td>0.127 0.245 0.475 0.153</td>
</tr>
<tr>
<td>$\bar{T}^T$</td>
<td>1.250 1.250 2.000 1.250</td>
<td>1.251 1.222 1.937 1.233</td>
<td>1.257 1.222 1.941 1.239</td>
</tr>
</tbody>
</table>

$\sqrt{\sum_{i,j} \hat{v}_{ij}/q^2} = 0.013$ $\sqrt{\sum_{i,j} \hat{v}_{ij}/q^2} = 0.009$ $\sqrt{\sum_{i,j} \hat{v}_{ij}/q^2} = 0.029$
Table VI. Occupation Times for Industrial DOF Data

<table>
<thead>
<tr>
<th></th>
<th>State = 1</th>
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<th>State = 3</th>
<th>State = 4</th>
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</thead>
<tbody>
<tr>
<td>DOF = -2</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Occupation C_i</td>
<td>3</td>
<td>1078</td>
<td>3</td>
<td>1101</td>
<td>3</td>
<td>1011</td>
</tr>
<tr>
<td>Time</td>
<td>1078</td>
<td>849</td>
<td>1974</td>
<td>2637</td>
<td>3459</td>
<td></td>
</tr>
<tr>
<td>Fractional F_i</td>
<td>-</td>
<td>0.108</td>
<td>-</td>
<td>0.11</td>
<td>-</td>
<td>0.085</td>
</tr>
<tr>
<td>Occupation time</td>
<td>0.108</td>
<td>0.085</td>
<td>0.197</td>
<td>0.264</td>
<td>0.346</td>
<td></td>
</tr>
<tr>
<td>Inverse of F_i</td>
<td>1</td>
<td>3333</td>
<td>1</td>
<td>3333</td>
<td>1</td>
<td>3333</td>
</tr>
<tr>
<td></td>
<td>9.3</td>
<td>11.8</td>
<td>5.1</td>
<td>3.8</td>
<td>2.9</td>
<td></td>
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Block A

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<tbody>
<tr>
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<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Occupation C_i</td>
<td>7</td>
<td>1076</td>
<td>7</td>
<td>1078</td>
<td>7</td>
<td>1011</td>
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<td>Time</td>
<td>1076</td>
<td>725</td>
<td>921</td>
<td>2396</td>
<td>4878</td>
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<tr>
<td>Fractional F_i</td>
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<td>-</td>
<td>0.108</td>
<td>-</td>
<td>0.073</td>
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<tr>
<td>Inverse of F_i</td>
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<td>2000</td>
<td>1</td>
<td>1250</td>
<td>1</td>
<td>1250</td>
</tr>
<tr>
<td></td>
<td>9.3</td>
<td>13.7</td>
<td>10.9</td>
<td>4.2</td>
<td>2.1</td>
<td></td>
</tr>
</tbody>
</table>

Block B

1\textsuperscript{st} entry in each row is for original data
2\textsuperscript{nd} entry in each row is for smoothed data

Table VII. Summary Statistics for Industrial DOF Data

<table>
<thead>
<tr>
<th></th>
<th>State = 1</th>
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<th>State = 3</th>
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<th>State = 5</th>
<th>State = 6</th>
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<tbody>
<tr>
<td>DOF = -2</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Transition</td>
<td>Mean=4.9 Median=2 Maximum=333 (State 6) Mean=8.9 Median=4 Maximum=334 (State 6)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Time Statistics</td>
<td>W</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean State</td>
<td>1</td>
<td>6.9</td>
<td>2.8</td>
<td>4.2</td>
<td>4.1</td>
<td>7.7</td>
</tr>
<tr>
<td>Holding Time</td>
<td>1</td>
<td>10.8</td>
<td>4.5</td>
<td>7.1</td>
<td>7.7</td>
<td>14.9</td>
</tr>
<tr>
<td>Mean Return Time</td>
<td>640.5</td>
<td>8.8</td>
<td>11.2</td>
<td>5.0</td>
<td>3.8</td>
<td>2.9</td>
</tr>
<tr>
<td>Time</td>
<td>640.5</td>
<td>8.6</td>
<td>11.7</td>
<td>5.0</td>
<td>3.8</td>
<td>2.9</td>
</tr>
<tr>
<td>Mean Transit</td>
<td>52.3</td>
<td>10.1</td>
<td>2.7</td>
<td>17.8</td>
<td>2.5</td>
<td>6.5</td>
</tr>
<tr>
<td>Time</td>
<td>70.3</td>
<td>13.2</td>
<td>3.6</td>
<td>29.5</td>
<td>2.9</td>
<td>8.5</td>
</tr>
</tbody>
</table>

Block A

<table>
<thead>
<tr>
<th></th>
<th>State = 1</th>
<th>State = 2</th>
<th>State = 3</th>
<th>State = 4</th>
<th>State = 5</th>
<th>State = 6</th>
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</thead>
<tbody>
<tr>
<td>DOF = -2</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>Transition</td>
<td>Mean=6.4 Median=2 Maximum=741(State 6) Mean=11.9 Median=4 Maximum=744 (State 6)</td>
<td></td>
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<td></td>
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<tr>
<td>Time Statistics</td>
<td>W</td>
<td></td>
<td></td>
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<tr>
<td>Mean State</td>
<td>2.3</td>
<td>12.1</td>
<td>3.9</td>
<td>2.6</td>
<td>4.3</td>
<td>13.5</td>
</tr>
<tr>
<td>Holding Time</td>
<td>4.0</td>
<td>18.9</td>
<td>6.5</td>
<td>4.6</td>
<td>8.2</td>
<td>26.9</td>
</tr>
<tr>
<td>Mean Return Time</td>
<td>1164</td>
<td>9.1</td>
<td>13.6</td>
<td>9.9</td>
<td>4.1</td>
<td>2.0</td>
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<tr>
<td>Time</td>
<td>997</td>
<td>9.1</td>
<td>13.5</td>
<td>10.0</td>
<td>4.1</td>
<td>2.1</td>
</tr>
<tr>
<td>Mean Transit</td>
<td>432</td>
<td>24.8</td>
<td>11.7</td>
<td>24.4</td>
<td>6.0</td>
<td>4.5</td>
</tr>
<tr>
<td>Time</td>
<td>432</td>
<td>35.4</td>
<td>18.1</td>
<td>42.6</td>
<td>9.7</td>
<td>8.3</td>
</tr>
</tbody>
</table>

Block B

1\textsuperscript{st} entry in each row is for original data
2\textsuperscript{nd} entry in each row is for smoothed data
Markov Chain Analysis of Industrial DOF Data (smoothed)

<table>
<thead>
<tr>
<th>$\hat{P}$</th>
<th>Block A</th>
<th>Block B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{P}$</td>
<td>0.000 0.667 0.333 0.000 0.000 0.000</td>
<td>0.750 0.125 0.000 0.125 0.000 0.000</td>
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<tr>
<td></td>
<td>0.003 0.908 0.063 0.013 0.008 0.005</td>
<td>0.002 0.948 0.028 0.013 0.007 0.002</td>
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<tr>
<td></td>
<td>0.000 0.084 0.779 0.102 0.021 0.014</td>
<td>0.000 0.064 0.847 0.060 0.022 0.007</td>
</tr>
<tr>
<td></td>
<td>0.000 0.012 0.045 0.860 0.067 0.017</td>
<td>0.000 0.005 0.062 0.782 0.123 0.272</td>
</tr>
<tr>
<td></td>
<td>0.000 0.001 0.005 0.055 0.871 0.068</td>
<td>0.000 0.002 0.008 0.051 0.879 0.061</td>
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<tr>
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<td>0.000 0.001 0.002 0.010 0.054 0.933</td>
<td>0.000 0.000 0.001 0.004 0.032 0.963</td>
</tr>
<tr>
<td>$\hat{T}$</td>
<td>0.000 0.110 0.081 0.198 0.267 0.344</td>
<td>0.001 0.110 0.073 0.091 0.242 0.483</td>
</tr>
<tr>
<td>$\sqrt{\sum_{i,j}^N \hat{V}_{ij}/q^2}$</td>
<td>0.090</td>
<td>0.070</td>
</tr>
</tbody>
</table>

Markov Chain Analysis of Compressed Industrial DOF Data (smoothed)

<table>
<thead>
<tr>
<th>$\hat{P}$</th>
<th>Block A</th>
<th>Block B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{P}$</td>
<td>0.910 0.063 0.263</td>
<td>0.949 0.028 0.234</td>
</tr>
<tr>
<td></td>
<td>0.084 0.779 0.137</td>
<td>0.064 0.847 0.089</td>
</tr>
<tr>
<td></td>
<td>0.004 0.013 0.983</td>
<td>0.001 0.010 0.989</td>
</tr>
<tr>
<td>$\hat{T}$</td>
<td>0.110 0.081 0.808</td>
<td>0.111 0.073 0.816</td>
</tr>
<tr>
<td>$\sqrt{\sum_{i,j}^N \hat{V}_{ij}/q^2}$</td>
<td>0.090</td>
<td>0.070</td>
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</tbody>
</table>

Markov Chain Analysis of Expanded Industrial DOF Data (smoothed)

<table>
<thead>
<tr>
<th>State</th>
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<th>$L_{t+1} L_t$</th>
<th>$L_{t+1} N_t$</th>
<th>$L_{t+1} H_t$</th>
<th>$N_{t+1} L_t$</th>
<th>$N_{t+1} N_t$</th>
<th>$N_{t+1} H_t$</th>
<th>$H_{t+1} L_t$</th>
<th>$H_{t+1} N_t$</th>
<th>$H_{t+1} H_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{P}_A$</td>
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<tr>
<td>$\hat{P}_A$</td>
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</tr>
<tr>
<td>$\hat{T}_A$</td>
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<tr>
<td>$\hat{T}_A$</td>
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</tbody>
</table>